In life insurance contracts, benefits and premiums are typically paid contingent on the biometric state of the insured. Due to delays between the occurrence, reporting, and settlement of changes to the biometric state, the state process is not fully observable in real-time. This fact implies that the classic multi-state models for the biometric state of the insured are not able to describe the development of the policy in real-time, which encompasses handling of incurred-but-not-reported and reported-but-not-settled claims. We give a fundamental treatment of the problem in the setting of continuous-time multi-state life insurance by introducing a new class of models: transaction time models. The relation between the transaction time model and the classic model is studied and a result linking the present values in the two models is derived. The results and their practical implications are illustrated for total permanent disability insurance, where we show how added structure allows one to obtain explicit expressions for the transaction time reserve.

Keywords: Prospective reserves; Disability insurance; Claims reserves; Valid and real-time; Piecewise deterministic processes.

2020 Mathematics Subject Classification: 91G05; 60J76.
JEL Classification: G22; C02.

1 Introduction

The payments stipulated in life insurance contracts are usually an agreement on what payments are to be made for different possible outcomes of the biometric state of the insured (e.g. whether the insured is active, disabled, dead, etc.). For this reason, multi-state life insurance models take modeling of the biometric state of the insured as their starting point. The multi-state approach to life insurance dates back to at least Hoem (1969). Here, the prospective reserve is defined as the discounted probability-weighted future payments, which, as noted in Norberg (1991), corresponds to the expected present value of future payments given the information generated by the biometric state process. The introduction of an underlying stochastic process generating the payments
introduces structure to the problem of predicting the cash flow at future points in time, due to the temporal dependencies of the process. This added structure of the payments is not in itself an assumption when the payments stipulated in the insurance contracts are formulated in terms of the biometric state of the insured. It is rather a way to introduce more a priori knowledge about the workings of the product into the mathematical model. All other things being equal, this makes the models more powerful.

Consequently, multi-state modeling seems a natural approach to modeling life insurance products. However, in the multi-state modeling literature, one also often assumes that the biometric state process generating the payments equals the process that generates the available information, see e.g. Norberg (1991), Buchardt et al. (2015), Djehiche and Löfdahl (2016), Bladt et al. (2020), and Christiansen (2021) to name a few. This is rarely the case, since information about changes to the biometric state can be delayed or erroneous. A simple example of this phenomenon is the delay that occurs when an insured becomes disabled; it might take some time for the insured to report the event to the insurer. Between the occurrence of the disability and the time of reporting, the claim is an IBNR (Incurred-But-Not-Reported) claim. As long as the insured has not reported the disability, the insurer will continue believing that the insured is active. Hence, the information that the insurer has is different from the full information about the biometric state of the insured. To describe this phenomenon in more detail, and discuss how to approach reserving under the insurer’s available information, we introduce the concepts of valid time and transaction time in the next section: Essentially, the valid time of an event is the time that it occurs, while the transaction time is the time that the event is registered in the insurers records.

It turns out that these concepts are also useful in clarifying the similarities and differences between life and non-life insurance products as well as between the models employed in the respective fields. The fact that payments in life insurance are deterministic functions of the biometric state process makes it so one does not have to estimate a separate distribution for the payment sizes; once the distribution of the state process is specified, the distribution of the payments follows. This is not the case in non-life insurance, and one therefore resorts to modeling the distribution of the observed payments directly. However, as will be explained, the biometric state process is a valid time object, while the observed payments are transaction time objects. This fact leads to key differences in the life and non-life insurance models. One such key difference is that it is more straightforward to formulate IBNR and RBNS (Reported-But-Not-Settled) models in non-life insurance, as one can construct all the relevant models entirely in transaction time. In life insurance models, one has to link the valid time payments to the transaction time concepts of IBNR and RBNS, and it is not obvious how to do this.

Our main contributions are the introduction of the basic bi-temporal structure assumptions defined in Section 4 and the derivation of a result linking the present values in valid and transaction time, which is presented in Theorem 5.4. The former constructs an explicit link between transaction time and valid time processes. The latter utilizes this link to obtain a simple relation between the present values in valid and transaction time. To obtain a simple relation between the valid time and transaction time reserves, more structure on how the transaction time information affects the distribution of the valid time process needs to be imposed. This is explored in a simple example. Together, these results allows one to link the valid time payments to the transaction time concepts of IBNR and RBNS.

The paper is structured as follows. In Section 2 the terms valid time and transaction time are given more precise definitions and discussed in the context of life insurance. An overview of the use of valid time and transaction time information in the insurance literature is provided, and similarities and differences between the situation in life insurance and non-life insurance are made explicit. The section ends by defining the class of piecewise deterministic processes, which
constitute the basic building blocks for our model constructions. Section 3 constructs a model for the insurance contract in valid time similarly to how the classic life insurance multi-state models are constructed. Section 4 introduces the novel concept of a transaction time model corresponding to a valid time model, and the transaction time model is constructed. In Section 5, the valid and transaction time reserves are defined, and the main result of this paper is derived, relating the transaction time present value to the valid time present value. In a model for total permanent disability insurance, we show how this relation can be utilized to obtain a relation between the corresponding reserves. Finally, the dynamics of the transaction time reserve is derived and discussed.

2 Valid and transaction time

We now introduce the terms valid time and transaction time. These concepts are used to describe data that arises from a time-varying process. We outline how these types of data are currently being used in the life and non-life insurance literature. Subsequently, we introduce a class of stochastic processes which we use to model processes generating valid time and transaction time data.

Valid and transaction time data

The terms valid time and transaction time provide a natural terminology for describing information that is registered with delays and uncertainty. Valid time and transaction time are concepts stemming from the design of databases, specifically temporal databases, where time-varying information is recorded. The valid/transaction time taxonomy was developed in Snodgrass and Ahn (1985). There, valid time is defined as the time that an event occurs in reality, while transaction time is defined as the time when the data concerning the event was stored in the database. Hence, valid time is concerned with when events occurred (historical information), while transaction time is concerned with when events were observed (rollback information).

As noted by Snodgrass and Ahn (1985), an important difference between valid time and transaction time are the types of information updates that are permitted. A transaction time may be added to the database, but is never allowed to be changed after the fact due to the forward motion of time. In contrast, a valid time is always subject to change, since discrepancies between the history as it actually occurred and the representation of the history as stored in the database will often be detected after the fact. The authors argue that both valid time and transaction time are needed to fully capture time-varying behavior.

A database that contains both valid time and transaction time is called a bi-temporal database. Such a database supplies both historical and rollback information. Historical information e.g. “Where was Taylor employed during 2010?” is supplied by valid time, while rollback information e.g. “In 2010, where did the database believe Taylor was employed?” is supplied by transaction time. Since there may have been changes to the database after 2010, the answers to these questions may be different. The combination of valid time and transaction time supplies information on the form “In 2015, where did the database believe Taylor was employed during 2010?”.

To further clarify the concepts introduced above, a detailed example in the context of total permanent disability insurance is given in Example 2.1.

Example 2.1. (Bi-temporal insurance data.)

Consider the following scenario: On 1/1/2020, Taylor buys a total permanent disability insurance effective immediately with a risk period of one year, which pays a sum b if they become disabled
before the end of the risk period. For this, Taylor agrees to pay premiums at a rate $\pi$ during the risk period while active. Taylor becomes disabled on 1/3/2020 and reports this to the insurer on 1/5/2020, two months later. On 1/6/2020, one month later, the insurer has finished processing the claim and awards Taylor disability benefits. Furthermore, Taylor is reimbursed for the premiums paid between 1/3/2020 and 1/6/2020.

If the insurer uses a bi-temporal database (valid time and transaction time), the database will at 1/6/2020 contain the following entries:

<table>
<thead>
<tr>
<th>Name</th>
<th>State</th>
<th>Valid From</th>
<th>Valid Till</th>
<th>Entered</th>
<th>Superseded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taylor</td>
<td>Active</td>
<td>1/1/2020</td>
<td>$\infty$</td>
<td>1/1/2020</td>
<td>1/5/2020</td>
</tr>
<tr>
<td>Taylor</td>
<td>Active</td>
<td>1/1/2020</td>
<td>1/3/2020</td>
<td>1/5/2020</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Taylor</td>
<td>RBNS</td>
<td>1/3/2020</td>
<td>$\infty$</td>
<td>1/5/2020</td>
<td>1/6/2020</td>
</tr>
<tr>
<td>Taylor</td>
<td>Disabled</td>
<td>1/3/2020</td>
<td>$\infty$</td>
<td>1/6/2020</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Hence we see that the database records not only what happened in the ‘real world’, but also what was officially recorded at different times. Note that when it is not known when the information is valid till, the database by convention records the timestamp $\infty$. This is likewise the case when it is unknown when the entry will be superseded. Hence, to acquire the most recent belief about when events occurred, one would extract the rows where Superseded was $\infty$.

If the database was uni-temporal (valid time), the entries at 1/6/2020 would be:

<table>
<thead>
<tr>
<th>Name</th>
<th>State</th>
<th>Valid From</th>
<th>Valid Till</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taylor</td>
<td>Active</td>
<td>1/1/2020</td>
<td>1/3/2020</td>
</tr>
<tr>
<td>Taylor</td>
<td>Disabled</td>
<td>1/3/2020</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Similarly, at 1/4/2020 the entries would be:

<table>
<thead>
<tr>
<th>Name</th>
<th>State</th>
<th>Valid From</th>
<th>Valid Till</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taylor</td>
<td>Active</td>
<td>1/1/2020</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

even though Taylor is already disabled at this time, due to the fact that this has not been reported to the insurer yet.

From these tables, we can see that, as described in Snodgrass and Ahn (1985), different information updates are permitted for a bi-temporal and a uni-temporal database. The uni-temporal database, in contrast to the bi-temporal database, only records what happened in the ‘real world’ based on the newest information. Previous records are modified or deleted.

Practitioners should be well-acquainted with bi-temporal insurance data. Bi-temporal data is important for internal use, as it is needed for reproducibility of statistical analyses when these are based on queries to a database. This is because reproducibility requires rollback information, since one has to recreate the information that the database had at a previous point in time. It also enables one to understand the difference between two otherwise identical analyses, performed at two different points in time. It is also important for external use, since auditors and regulatory authorities may inquire about financial reports from foregone years, making it important for insurers to be able to recreate the prerequisites that a given financial report was based on. As an example of this, Danish life insurance companies are required by law to publish all figures in the balance sheet and income statement of their financial reports for both the current and the previous year. Key figures have to be reported for the past five years. If prior financial reports have been
affected by serious errors, the newest report has to publish figures for previous years as if the error had not been committed, so long it is practically feasible, cf. § 86 of [Erhvervsministeriet] (2015).

**Valid and transaction time information in insurance**

Inspired by the valid and transaction time taxonomy introduced above, and with a slight abuse of the terminology, we define a **valid time process** as a stochastic process that represents the true historical information. We use the notation $X_s$ for the value of the valid time process at time $s$. With another slight abuse of the terminology, we define a **transaction time process** as a bi-temporal stochastic process that represents both historical and rollback information. We use the notation $X_t^s$ for the value of the process at time $s$ based on the observations available at time $t$. When referring to models based on valid time and transaction time processes, we use the terms **valid time model** and **transaction time model**, respectively. Current multi-state models for the biometric state of the insured are seen to be valid time models, as they model a process that describes when events occur without regard to when that information is observed by the insurer. However, in practice there is typically at least some delay in information concerning occurrence, reporting and settlement of claims. This necessitates additional model components, say in the form of models for IBNR and RBNS claims. Such model components are called claims reserving models or simply IBNR and RBNS models.

Claims reserving based on individual claims data have been subject to much study in non-life insurance, see e.g. [Norberg (1993, 1999)], [Haastrup and Arjas (1996)], [Antonio and Plat (2014)], [Badescu et al. (2016, 2019)], [Lopez et al. (2019)], [Bischofberger et al. (2020)], [Delong et al. (2021)], [Crevecoeur et al. (2022)], and [Okine et al. (2022)]. Early research, including [Norberg (1993, 1999)], focuses on joint modeling of all aspects of a claim and the subsequent computation of relevant conditional expectations of future cash flows as high-dimensional integrals with respect to the joint distribution. This may pose significant statistical and numerical challenges, and initially only limited attention was given to practical implementation. Recent research, including [Crevecoeur et al. (2022)] and [Okine et al. (2022)], instead focuses on how the expected future cash flows depend on previous observations, typically payment times and payment sizes, and the corresponding factorization of the joint distribution into conditional and marginal parts. In particular, this improves interpretability and readily allows for dynamic reserving where current information is incorporated into the best estimate of future liabilities. This research has hitherto largely consisted of data-driven investigations. Here one attempts to identify the best predictive models by exploring which covariates are advantageous to include (thereby also determining the degree of individualization/collectivization) as well as exploring which statistical models to apply (e.g. parametric models such as GLMs, non-parametric models such as empirical distributions, or machine learning methods such as neural networks).

The primary reason that the life insurance literature has not been similarly occupied with finding suitable statistical models is, as explained in Section II that the payments stipulated in life insurance contracts are specified in terms of an underlying valid time state process. This implies that the conditional distribution of future jump times given previous observations can be obtained by estimating the distribution of the valid time state process, and a model for the conditional distribution of future payment sizes then follows automatically. This also allows for more explanatory models compared with the purely predictive models of non-life insurance that result from deciding to model the transaction time payment process directly.

The problem with the current approach in life insurance of ignoring the transaction time state process is that the alternative state process, namely the valid time state process, is not directly observable. In practice, one partly accounts for this via improvised IBNR corrections and RBNS
corrections (on an aggregate level). Compared to the vast literature on IBNR and RBNS models in non-life insurance, the corresponding problem of claims reserving in life insurance has received limited attention. In the following, we seek to amend this.

Since the payments stipulated in a life insurance contract are generated by a valid time process, while the actually observed data is generated by a transaction time process, it becomes essential to establish an in-depth understanding of the relation between valid and transaction time processes in life insurance. The main purpose of this paper is to develop a conceptual and mathematical framework where this relation can be formulated and explored.

**Piecewise deterministic processes**

In order to achieve as much generality as possible, we take piecewise deterministic processes (PDPs) as the starting point for our models. The data generated by a stochastic process and recorded in a database only uniquely determines the path of the stochastic process if the stochastic process is a PDP. Informally, these are processes that have finitely many jumps on finite time intervals and which develop deterministically between the random jump times. This is because the value of the process is only recorded in the database at certain discrete times (e.g. daily, monthly, or at jump times), from which the entire path of the process must be inferred. If the data stems from e.g. a Brownian motion, which is not piecewise deterministic, then the data can only provide an approximation of the path taken, and the gaps between recorded values must be filled using some algorithm, e.g. modeling the process to be linearly evolving between recorded values. In this sense, PDPs provide the most general class of processes for which valid time and transaction time processes can be constructed.

We now define piecewise deterministic processes following Subsection 3.3 in [Jacobsen (2006)]. Let the background probability space be denoted by \((\Omega, \mathcal{F}, \mathbb{P})\). Let \((E, \mathcal{E})\) be a measurable space called the **mark space**. Introduce the **irrelevant** mark \(\nabla\) denoting the mark of a jump that does not occur in finite time, and set \(\overline{E} = E \cup \{\nabla\}\).

**Definition 2.2.** (Simple point processes.)

A **simple point process** (SPP) is a sequence \(T = (T_n)_{n \in \mathbb{N}}\) of \([0, \infty)\)-valued random variables defined on \((\Omega, \mathcal{F}, \mathbb{P})\) such that

(i) \(\mathbb{P}(0 < T_1 \leq T_2 \leq \ldots) = 1\)

(ii) \(\mathbb{P}(T_n < T_{n+1}, T_n < \infty) = \mathbb{P}(T_n < \infty)\)

(iii) \(\mathbb{P}(\lim_{n \to \infty} T_n = \infty) = 1\). \(\triangle\)

If condition (iii) is removed, one obtains the class of simple point processes allowing for **explosion**. In this paper, we limit the study to processes without explosion.

**Definition 2.3.** (Marked point processes.)

A **marked point process** (MPP) with mark space \(E\) is a double-sequence \((T, Y) = ((T_n)_{n \in \mathbb{N}}, (Y_n)_{n \in \mathbb{N}})\) of \([0, \infty)\)-valued random variables \(T_n\) and \(\overline{E}\)-valued random variable \(Y_n\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\) such that \(T = (T_n)_{n \in \mathbb{N}}\) is an SPP and such that

(i) \(\mathbb{P}(Y_n \in E, T_n < \infty) = \mathbb{P}(T_n < \infty)\)

(ii) \(\mathbb{P}(Y_n = \nabla, T_n = \infty) = \mathbb{P}(T_n = \infty)\). \(\triangle\)
For a given MPP \((T, \mathcal{Y})\), define

\[
\langle t \rangle = \sum_{n=1}^{\infty} 1(T_n \leq t)
\]

being the number of events in the time interval \([0, t]\), and define

\[
H_t = (T_1, ..., T_{\langle t \rangle}; Y_1, ..., Y_{\langle t \rangle})
\]

being the jump times and marks observed up until and including time \(t\). We refer to \(H_t\) as the MPP history at time \(t\).

**Definition 2.4.** (Piecewise deterministic process.)

A piecewise deterministic process with state space \((E, \mathcal{E})\) is an \(E\)-valued stochastic process \(X\) satisfying

\[
X_t = f^{(t)}_{H_t|x_0}(t),
\]

where \(X_0 = x_0\) is non-random, and for every \(n \in \mathbb{N}_0\), \(f^n_{h_n|x_0}(t)\) is a measurable \(E\)-valued function of \(h_n = (t_1, ..., t_n; y_1, ..., y_n)\) with \(t_n < \infty\), of \(t \geq t_n\), and of \(x_0\), satisfying the conditions

\[
f^n_{h_n|x_0}(t_n) = y_n
\]

for all \(h_n\), and \(f^0_{x_0}(0) = x_0\).

To ensure the existence of relevant conditional distributions going forward, we henceforth assume that the mark space \((E, \mathcal{E})\) is a Borel space. We refer to the functions \(f^n_{h_n|x_0}\) as evolution functions of \(X\). It is easily seen that the class of piecewise deterministic processes encompasses the usual choices for the biometric state process, cf. Example 2.5, 2.6, 2.7, and 2.8.

**Example 2.5.** (Pure jump process.)

Let \((E, \mathcal{E})\) be a Borel space, and let \(X\) be a pure jump process taking values in \(E\) and modeling the biometric state process of the insured. Assume that the initial state of \(X\) is fixed, and denote this initial state by \(x_0\). A pure jump process is here taken to mean a càdlàg stochastic process with only finitely many jumps in any finite time interval satisfying

\[
X_t = \sum_{0 < s \leq t} \Delta X_s
\]

for \(\Delta X_s = X_s - X_{s-}\). We define an MPP \((T, \mathcal{Y}) = ((T_n)_{n \in \mathbb{N}}, (Y_n)_{n \in \mathbb{N}})\) from \(X\) by letting \(T_n\) be the time of the \(n\)’th jump of \(X\) and setting \(Y_n = X_{T_n}\). It is then easy to show that \(X\) can be reconstructed from the MPP. To do this, let \(f^n_{h_n|x_0}(t) = y_n\), and let \(H\) be the MPP history of \((T, \mathcal{Y})\). Then

\[
f^{(t)}_{H_t|x_0}(t) = Y_{\langle t \rangle} = X_{T_{\langle t \rangle}} = X_t,
\]

so \(X\) is a PDP.

**Example 2.6.** (Pure jump Markov process.)

Following the previous example, let \(\mathcal{F}^X_t = \sigma(X_s, 0 \leq s \leq t)\) be the filtration generated by \(X\).
For computational tractability, one often assumes that $X$ is a \textit{Markov process}, meaning for every $s \leq t$ and $C \in \mathcal{E}$:

$$\mathbb{P}(X_t \in C \mid \mathcal{F}_s^X) = \mathbb{P}(X_t \in C \mid X_s)$$

\textit{P}-a.s., such that the behavior of the process at a future time point is only dependent on the past behavior of the process through the current state of the process. This is called the \textit{Markov property}. Pure jump Markov processes $X$ are a generalization of the usual choices of the state process $Z$ found in the multi-state life insurance literature, as is shown in Example 2.7 and Example 2.8.

\textbf{Example 2.7.} (Continuous-time Markov chain.)

A continuous-time Markov chain $Z$ with fixed initial state is defined as a piecewise constant Markov process, see e.g. Chapter 2 of Norris (1998) or Section 7.2 of Jacobsen (2006). In the life insurance literature, the Markov chain is usually assumed to take values in a finite state space, see e.g. Norberg (1991). A continuous-time Markov chain $Z$ on a finite state space $\{1, 2, ..., J\}$ for $J \in \mathbb{N}$ can thus be constructed from a pure jump Markov process $X$ with $E = \{1, 2, ..., J\}$ by simply setting $Z = X$.

\textbf{Example 2.8.} (Continuous-time semi-Markov process.)

In many applications, the most recent jump time contains valuable information and hence it may be necessary to include it as a coordinate of $X$ in order to ensure that the Markov property of $X$ holds. For a pure jump process $Z$, define $W$ as the time of the last jump $W_t = \sup\{0 \leq s \leq t : Z_s \neq Z_t\}$.

A continuous-time semi-Markov process $Z$ is defined as a pure jump process on a finite state space with the property that $(Z, W)$, equivalently $(Z, U)$ with $U_t = t - W_t$ the duration of sojourn in the current state, is a Markov process, see e.g. Section 2D in Helwich (2008). Semi-Markov models were introduced to life insurance independently by Janssen and Hoem, see Janssen (1966) and Hoem (1972). In the life insurance literature, the parameterization $(Z, U)$ is more common than $(Z, W)$.

For $K \in \mathbb{N}$ define the projection functions $\pi_k : \mathbb{R}^K \mapsto \mathbb{R}$ via $\pi_k(x_1, x_2, ..., x_K) = x_k$. If $X$ is a pure jump Markov process with $E = \{1, 2, ..., J\} \times [0, \infty)$ that satisfies $\pi_2 X_t = T(t)$, then a continuous-time semi-Markov process $Z$ may be constructed from $X$ by setting $Z = \pi_1 X$.

\section{3 Valid time model}

We now introduce the classic multi-state life insurance models. These are valid time models, cf. the discussion in Section 2. The valid time setup described below is essentially standard in the multi-state life insurance literature, although it is usually only formulated for Markov or semi-Markov processes, see the classics Hoem (1969), Hoem (1972), and Norberg (1991).

\textbf{State process}

Let the valid time state process $(X_t)_{t \geq 0}$ be an $\mathbb{R}^d$-valued stochastic process for $d \in \mathbb{N}$. We assume that $X$ is a PDP with fixed initial state $x_0$. The jump times are denoted by $\tau_n$. Let

$$\langle t \rangle = \sum_{n=1}^{\infty} 1_{(\tau_n \leq t)}$$
denote the number of jumps by time \( t \) and denote by

\[
H_t = (\tau_1, \ldots, \tau_{\langle t \rangle}; X_{\tau_1}, \ldots, X_{\tau_{\langle t \rangle}})
\]

the MPP history of the process \( X \) at time \( t \). The symbols \( \tau_n \) and \( \langle t \rangle \) are used here instead of \( T_n \) and \( (t) \) introduced in the PDP section as the latter are reserved for the transaction time process introduced in Section 4. Further, let \( f^n_{h_n|x_0} \) be the evolution functions of \( X \), such that

\[
X_t = f^{\langle t \rangle}_{H|\{x_0 \} \}(t).
\]

The information generated by complete observation of the valid time process is given by the filtration

\[
F^X_t = \sigma(X_s, 0 \leq s \leq t).
\]

**Cash flow**

Write \( x_A \) for \((x_s)_{s \in A} \) with \( A \subseteq [0, \infty) \), where \( x_A \) is such that \((x_s)_{s \geq 0} \) lies in the support of \( X \) for some choice of \((x_s)_{s \in A'} \). Similarly, write \( X_A \) for \((X_s)_{s \in A} \). Assume the existence of measurable functions

\[
(x_A, t) \mapsto B(x_A, t) \in \mathbb{R}
\]

for \( t \geq 0 \). We interpret \( B(x_A, t) \) as the payments generated by the path \( x \) on \( A \cap [0, t] \). Assume that \( t \mapsto B(x_A, t) \) is a càdlàg finite variation function for any \( x_A \), so that the measures \( B(x_A, dt) \) are well-defined. We further assume that the composition \( B(X_A, t) \) is incrementally adapted to \( F^X_t \), meaning that \( B(X_A, t) - B(X_A, s) \) is \( \sigma(X_v, v \in (s, t]) \)-measurable for any interval \((s, t] \subseteq [0, \infty) \), cf. Definition 2.1 in [Christiansen (2021)]. For shorthand, we write \( B(dt) = B(X_{[0, \infty)}, dt) \). The assumption of incrementally adaptedness states that the aggregated payments in \((s, t]\) only depends on the path of \( X \) on \((s, t]\), which informally may be written as \( B(dt) = B^{X_s - X_t}(dt) \). This informal notation is revisited in the definition of the transaction time cash flow. We name \((B(t))_{t \geq 0} \) the accumulated cash flow in valid time. Note the use of the symbol \( B \) for two different objects, namely the stochastic process \( t \mapsto B(t) \) and the deterministic function \( t \mapsto B(x_A, t) \). It should always be clear from the context which object we are referring to.

To allow for the valuation of cash flows, we need the time value of money. A detailed treatment of this financial constituent of the model may be found in [Norberg (1990)]. Let \( t \mapsto \kappa(t) \) be some deterministic strictly positive càdlàg accumulated function with initial value \( \kappa(0) = 1 \). The corresponding discount function is \( t \mapsto \frac{1}{\kappa(t)} \). We let \( x_A \mapsto B^\kappa(x_A) \) be the time 0 value of the accumulated payments for the path \( x_A \), i.e. we set

\[
B^\kappa(x_A) = \int_{(0, \infty)} \frac{1}{\kappa(v)} B(x_A, dv),
\]

presupposing that this object exists (is finite).

**Remark 3.1.** (Cash flow terminology.)

In the life insurance literature, the stochastic process \((B(t))_{t \geq 0} \) defined above is sometimes also referred to as the payment function, the stream of net payments, the payment process, or the stochastic cash flow, see e.g. [Norberg (1991)] and [Buchardt et al.] (2015). We use the terminology that \( B(t) \) is the accumulated cash flow at time \( t \) while the stochastic measure \( B(dt) \) is the cash flow. The latter should not be confused with the expected cash flow \( A_x(t, ds) \) defined by

\[
A_x(t, s) = \mathbb{E}[B(s) - B(t) \mid X(t) = x], \ s \geq t.
\]

In the literature, the expected cash flow is sometimes also ambiguously referred to as the cash flow.
Example 3.2. (Cash flow for the usual choices of state processes.)
For a set $A$ on the form $[0, v], [0, v], \text{ or } [0, \infty)$ for $v \geq 0$ and a pure jump Markov process $X$ on a
finite state space $E = \{1, 2, \ldots, J\}$, see also Example 2.7, one usually specifies the payments as
\[
B(x_A, dt) = \sum_{j=1}^{J} 1_{A(t)}(x_t = j) B_j(dt) + \sum_{j,k=1}^{J} 1_{A(t)}(t) b_{jk}(t) n_{jk}(x_A, dt),
\]
where $n_{jk}(x_A, t) = \#\{s \in [0, t] \cap A : x_s = j, x_s = k\}$, while $t \mapsto B_j(t)$ are càdlàg finite variation functions modeling sojourn payments and $t \mapsto b_{jk}(t)$ are finite-valued Borel-measurable functions modeling transition payments. In this case,
\[
B(dt) = \sum_{j=1}^{J} 1(x_t = j) B_j(dt) + \sum_{j,k=1}^{J} 1_{A(t)}(t) b_{jk}(t) N_{jk}(dt)
\]
for $N_{jk}(t) = n_{jk}(X_{[0,\infty)}, t)$. In the semi-Markov case of Example 2.8, where $X = (Z, W)$ is a pure Markov jump process (with $W$ the time of the last jump), one usually specifies the payments as
\[
B(x_A, dt) = \sum_{j=1}^{J} 1_{A(t)}(z_t = j) B_j,w(t) dt + \sum_{j,k=1}^{J} 1_{A(t)}(t) b_{j,w}(k, t) n_{jk}(z_A, dt),
\]
where $x_s = (z_s, w_s)$, while $t \mapsto B_j,w(t)$ and $t \mapsto b_{j,w}(k, t)$ satisfy the same regularity conditions as in the Markov case. Then
\[
B(dt) = \sum_{j=1}^{J} 1(Z_t = j) B_j,W(dt) + \sum_{j,k=1}^{J} 1_{A(t)}(t) b_{j,W}(k, t) N_{jk}(dt)
\]
for $N_{jk}(t) = n_{jk}(Z_{[0,\infty)}, t)$.

Example 3.3. (Continuous compound interest.)
Under continuous compound interest with force of interest $t \mapsto r(t)$, a deposit of one unit currency in a savings account at time 0 has at time $t$ accumulated to
\[
\kappa(t) = \exp \left( \int_{(0,t)} r(s) \, ds \right),
\]
see e.g. Norberg (1990).

Example 3.4. (Total permanent disability: Valid time model.)
We construct a valid time model for the product described in Example 2.1. The state of the insured is modeled as a pure jump process $X$ on the state space depicted in Figure 3.1.

Figure 3.1: The biometric state process $X$ takes values in $\{a, i, d\}$. The absence of an arrow between states indicates that a direct jump between these states is impossible.
Observe that $X$ is a PDP by Example 2.5. The cash flow is assumed to consist of a risk period $\rho > 0$, a premium rate $\pi < 0$ while active, and a disability sum $b > 0$ upon disability,

$$B(x_A, dt) = 1_A(t)\pi 1_{[0,\rho]}(t) 1_{(x_t = a)} \, dt + 1_A(t)b 1_{[0,\rho]}(t) n_{ai}(x_A, dt)$$

such that

$$B(dt) = \pi 1_{[0,\rho]}(t) 1_{(X_t = a)} \, dt + b 1_{[0,\rho]}(t) N_{ai}(dt).$$

Note that because of the payment structure, we can omit the modeling of death after disability, as there are no payments strictly after having entered the disabled state. The model introduced above could also be used to model critical illness products by replacing the disability state by a critically ill state. We revisit and extend this example in Examples 4.3 and 5.7 below.

4 Transaction time model

We now introduce the transaction time models corresponding to the valid time models introduced in Section 3. The transaction time setup described below is novel, and it allows for cash flows that are tailored to describe the payments that occur in real-time, like the ones that would result from the case described in Example 2.1.

State process

Let $(Z_t)_{t \geq 0}$ be an $\mathbb{R}^q$-valued pure jump process for $q \in \mathbb{N}$. We think of $Z$ as describing the claim settlement process of the policy, which is observed in real-time by the insurer. The jump times are denoted by $T_n$. We write

$$\langle t \rangle = \sum_{n=1}^{\infty} 1_{(T_n \leq t)}$$

for the number of jumps of $Z$ that have happened at time $t$.

We introduce a doubly indexed stochastic process $H^t_s$ for $0 \leq s \leq t$, which we name the transaction time MPP history corresponding to $H_s$. The idea is to interpret $H^t_s$ as the value of $H_s$ based on the transaction time information available at time $t$. We extend the definition to $s > t$ by letting $H^t_s = H^t_t$, akin to how in Example 2.1 the most recent history is set to be valid until $\infty$. To imbue $H^t_s$ and $Z$ with the desired interpretations outlined above, we specify the dependencies to the corresponding valid time process:

(i) We only allow changes to the transaction time MPP history corresponding to $H_s$ to occur at jumps of $Z$, so the process $Z$ is the driver of new transaction time information arriving. This is formalized as

$$H^t_s = H^T_{s(t)}$$

for all $t, s \geq 0$.

(ii) We assume that there is a finite time after which no new information arrives (e.g. the time of death of the insured). This is formalized as

$$Z_t = Z_\eta$$

for all $t \geq \eta$ with $\mathbb{P}(\eta < \infty) = 1$. This condition is satisfied in any practical application. We name $\eta$ the absorption time.
(iii) When there are no future changes to the transaction time MPP history corresponding to $H_s$, the observations are taken to be the true historical information. These observations constitute the finalized timeline that the insurer will observe. This is formalized as

$$H_s^t = H_s$$

for all $t \geq \eta$ and $s \geq 0$. This is the fundamental link between the valid time and transaction time models.

We name these assumptions the \textit{basic bi-temporal structure assumptions}.

Let $\ell_s^t$ denote the number of jumps contained in $H_s^t$. The transaction time state process is then defined as the doubly indexed stochastic process $X_s^t$ satisfying

$$X_s^t = \ell_s^t \left|_{H_s^t \mid x_0} \right. (s),$$

with the interpretation that $X_s^t$ is the value of $X_s$ based on the available transaction time information at time $t$.

At time $t$, the insurer has observed $H_s^t$ and $Z_s$ for all $s \leq t$. The insurer’s available information is therefore generated by a process $(Z_t)_{t \geq 0}$ given by

$$t \mapsto Z_t = (Z_t, H_t^t),$$

and we define the \textit{transaction time information} to be the filtration $\mathcal{F}_t^Z = \sigma(Z_s, 0 \leq s \leq t)$. Note that $Z$ is a PDM by Example 2.5, since it is a pure jump process which furthermore can be embedded into the Borel space $(\mathbb{R}\infty, \mathcal{B}(\mathbb{R}\infty))$.

\textbf{Remark 4.1.} (Transaction and valid time filtrations.) Note that in general, it holds that $\mathcal{F}_t^Z \not\subseteq \mathcal{F}_t^X$ and $\mathcal{F}_t^X \not\subseteq \mathcal{F}_t^Z$. This corresponds to $(Z_s)_{0 \leq s \leq t}$ not being known from $(X_s)_{0 \leq s \leq t}$ and vice versa. In other words, the same valid time realizations can stem from different transaction time realizations, and transaction time realizations in a period do not generally determine the valid time realizations in that same period, due to the possibility of new information arriving later.

\textbf{Cash flow}

We denote the accumulated cash flow in transaction time by $(\mathcal{B}(t))_{t \geq 0}$ and define it as

$$\mathcal{B}(dt) = B_{X_{t-}^t \rightarrow X_t^t}(dt) + d \left( \sum_{0 < s \leq t} \kappa(s) \left( B^{o}(X_s^{t \uparrow} - X_s^{t \downarrow}) \right) \right), \quad \mathcal{B}(0) = B(0),$$

where $B_{X_{t-}^t \rightarrow X_t^t}(dt)$ is informal notation for

$$\sum_{n=0}^{\infty} 1_{(t)=n} B(X_{[0,\infty)}^{T_{n}}), \quad dt$$

with the convention $T_0 = 0$. The notation $B_{X_{t-}^t \rightarrow X_t^t}(dt)$ is chosen to emphasize the relation to the valid time cash flow $B_{X_{t-}^t \rightarrow X_t^t}(dt)$. Hence the cash flow in transaction time consists of running payments similar to the ones in the valid time model, but here determined by the value of $X_{t-}^t$ and $X_t^t$ instead of $X_{t-}$ and $X_t$, as well as lump sum payments when the some past $X$ values
are changed based on the newest transaction time information. The lump sum payments are commonly known as backpay. Backpay makes the accumulated historic payments congruent with the latest MPP history. Furthermore, and closely connected to the principle of no arbitrage, the backpay is accumulated to the time of payout according to $\kappa$, so that the insured is no better or worse off than if the payment had not been delayed and had been deposited in a savings account immediately after payout.

Note that since the first $B$-term is evaluated in $X^t$ and not $X^{t-}$, the payment at time $t$ based on the most recent information is included in the first $B$-term and should not be included in the backpay, which is why the right endpoint is excluded in the interval $[0, s)$, that appears in the second term. Note also that $X^s_{[0,s)} = X^{s-}_{[0,s)}$ unless $Z$ jumps at time $s$. This implies that backpay can only be paid at jumps of $Z$.

**Remark 4.2.** (Cash flow modeling in non-life insurance.)

As described in Section 2 using non-life insurance methods, one would model the expected future payments arising from the real-time cash flow $B$ by disregarding the state process $X^t_s$ and the structure it imposes on the cash flow, and instead find suitable statistical models that predict $B(d_t)$ directly.

**Example 4.3.** (Total permanent disability: Transaction time model.)

We here describe a possible transaction time model corresponding to the valid time model from Example 3.4; it is implicitly presupposed that the bi-temporal structure assumptions are satisfied. The claim settlement process is here modeled as a pure jump process $Z = (Z^{(1)}, Z^{(2)})$ where the coordinate $Z^{(1)}$ takes values in the state space in Figure 4.1.

Figure 4.1: The claim settlement process $Z^{(1)}$ takes values in $\{1, 2, 3, 4, 5\}$. The absence of an arrow between states indicates that a direct jump between these states is impossible.

The coordinate $Z^{(2)}$ takes values in $[0, \infty)$ and represents the time of the disability reported by the insured in connection with a claim. Let $T^j = \inf\{s \geq 0 : Z^{(1)}_s = j\}$. We assume that $Z^{(1)}$ is absorbed in state 4 or 5, i.e. $\eta = \min\{T^4, T^5\}$, meaning that the insured becomes disabled or dies eventually. A jump of $Z^{(1)}$ from 2 to 4 is interpreted as an award of disability benefits from time $Z^{(2)}$, a jump from 2 to 3 is interpreted as a rejection of the disability benefits application, while a jump from 3 to 2 is interpreted as a reapplication. We then specify $H^t_s$ as follows: $H^t_s$ contains $(Z^{(2)}_{T^4}, i)$ or $(T^5, d)$ if $(T^4 \leq t, Z^{(2)}_{T^4} \leq s)$ or $(T^5 \leq t, T^5 \leq s)$, respectively, and it is empty otherwise. This means that there is a delay of $T^4 - Z^{(2)}_{T^4}$ between the disability occurrence and the award of benefits, while notification about death is instant. Then, by composition with $f$,

$$X^t_s = f^{(s)}_{H^t_s[x_0]}(s) = \begin{cases} 
  d, & T^5 \leq t, T^5 \leq s \\
  i, & T^4 \leq t, Z^{(2)}_{T^4} \leq s \\
  a, & \text{otherwise}.
\end{cases}$$
We have now established a model for $Z_t = (Z_t, H_t^1)$. Using the valid time cash flow from Example 3.4 we see that the transaction time cash flow $B(dt)$ reads

$$B(dt) = B^{X_{t^-}X_t}(dt) + \left( \sum_{0<s\leq t} \kappa(s) \left( B^0(X_{[0,s]}) - B^0(X_{[0,s]^-}) \right) \right)$$

$$= \pi 1_{[0,\rho]}(t) 1_{(X_{t} = a)} dt + \kappa(t) \left( \int_{[0,Z^{(2)}_t]} \frac{1}{\kappa(s)} \pi 1_{[0,\rho]}(s) ds + \frac{1}{\kappa(Z^{(2)}_t)} b 1_{[0,\rho]}(Z^{(2)}_t) \right)$$

$$- \int_{[0,t]} \frac{1}{\kappa(s)} \pi 1_{[0,\rho]}(s) ds \right) \left( d(1_{(Z^{(1)}_t = 4)}) \right)$$

$$= \pi 1_{[0,\rho]}(t) 1_{(Z^{(1)}_t \in \{1,2,3\})} dt + \left( \frac{\kappa(t)}{\kappa(Z^{(2)}_t)} b 1_{[0,\rho]}(Z^{(2)}_t) \right)$$

$$- \int_{[Z^{(2)}_t, t]} \frac{\kappa(t)}{\kappa(s)} \pi 1_{[0,\rho]}(s) ds \right) \left( d(1_{(Z^{(1)}_t = 4)}) \right),$$

such that premiums are paid continuously as long as death or a disability award has not been observed. The disability benefit $b$ is paid out at time $t$ if $Z^{(1)}_t$ jumps to state 4, but it is accumulated with interest from when the disability occurred $Z^{(2)}_t$ to the time of payout $t$. Likewise, the premiums that the insured had paid in the period $[Z^{(2)}_t, t]$ are accumulated with interest and paid back to the insured.

5 Reserving

Having defined the state process and cash flow in the valid and transaction time models, we are now in a position to define the prospective present value and prospective reserve in the respective models. The main result of the paper, Theorem 5.4, linking the present values in the two models, is deduced and discussed. Finally, the dynamics of the transaction time reserve is derived and its role in model validation is briefly considered.

Valid time reserve

Let the classic valid time prospective present value $(P(t))_{t \geq 0}$ be defined by

$$P(t) = \int_{(t,\infty)} \frac{\kappa(t)}{\kappa(s)} B(ds).$$

We assume that $P(t)$ is well-defined and belongs to $L^1(\Omega, \mathcal{F}, \mathbb{P})$ for every $t \geq 0$. Then we can define the corresponding expected present value by

$$V(t) = \mathbb{E}[P(t) | \mathcal{F}^X_t] \quad (5.1)$$

for any $t \geq 0$. As a function of $t$, this is also known as the prospective reserve in valid time. We consider a version of $V$ presumed to be $\mathcal{F}^X$-adapted and $\mathbb{P}$-a.s. right-continuous.

Remark 5.1. (Reserve for Markov state process.)

Note that $P(t)$ is $\sigma(X_s, t \leq s < \infty)$-measurable, so if $X$ is a Markov-process, it follows that

$$V(t) = \mathbb{E}[P(t) | X_t].$$

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For random variables $Y_1$ and $Y_2$, we say that $Y_1 = Y_2$ on $A \in \mathcal{F}$ if $Y_1(\omega) = Y_2(\omega)$ for $\mathbb{P}$-almost all $\omega \in A$. We immediately have the following representation of the present value:

**Proposition 5.2. (Valid time present value.)**

For $t \geq 0$, it holds that

$$P(t) = \kappa(t) \left(B^\circ(X^\eta_{[0,\infty)}) - B^\circ(X^\eta_{[0,t]})\right)$$

on the event $(\eta < \infty)$.

**Proof.** Using the definitions introduced above, we find that

$$P(t) = \kappa(t) \int_{(t,\infty)} \frac{1}{\kappa(s)} B(ds)$$

$$= \kappa(t) \left(\int_{[0,\infty)} \frac{1}{\kappa(s)} B(ds) - \int_{[0,t]} \frac{1}{\kappa(s)} B(ds)\right)$$

$$= \kappa(t) \left(\int_{[0,\infty)} \frac{1}{\kappa(s)} B(X_{[0,\infty)}, ds) - \int_{[0,t]} \frac{1}{\kappa(s)} B(X_{[0,\infty)}, ds)\right)$$

$$= \kappa(t) \left(B^\circ(X_{[0,\infty)}) - B^\circ(X_{[0,t]})\right).$$

Now on the event $(\eta < \infty)$ we have that $X_s = X^\eta_s$ for all $s \geq 0$, from which the result follows. 

**Transaction time reserve**

The transaction time prospective present value $(\mathcal{P}(t))_{t \geq 0}$ is defined as

$$\mathcal{P}(t) = \int_{(t,\infty)} \frac{\kappa(t)}{\kappa(s)} B(ds).$$

We assume $\mathcal{P}(t)$ is well-defined and belongs to $L^1(\Omega, \mathcal{F}, \mathbb{P})$ for any $t \geq 0$. Let $\mathcal{V}$ be the corresponding expected present value in the transaction time model

$$\mathcal{V}(t) = \mathbb{E}[\mathcal{P}(t) \mid \mathcal{F}^\mathcal{Z}_t]. \quad (5.2)$$

Just as for $V$, we consider a version of $\mathcal{V}$ presumed to be $\mathcal{F}^\mathcal{Z}$-adapted and $\mathbb{P}$-a.s. right-continuous. Since the backpay $B^\circ(X^\eta_{[0,s)}) - B^\circ(X^\eta_{[0,s]})$ telescopes, we also get the following representation of the transaction time present value $\mathcal{P}$:

**Proposition 5.3. (Transaction time present value.)**

For $t \geq 0$, it holds that

$$\mathcal{P}(t) = \kappa(t) \left(B^\circ(X^\eta_{[0,\infty)}) - B^\circ(X^\eta_{[0,t]})\right)$$

on the event $(\eta < \infty)$.

**Proof.** Note that the equality is trivially satisfied on $(t \geq \eta)$ by Proposition 5.2 and the observation that $P(t) = \mathcal{P}(t)$ on $(t \geq \eta)$ since $X^t = X^\eta$ implies that the running payments agree and there is no backpay after time $t$. Hence, what remains to be shown is that the equality also holds on
(t < \eta). We therefore let all the remaining calculations be on (t < \eta). We re-index \( T_k = T_{k+t} \) for \( k \in \mathbb{N} \), so \( T_k \) now refers to the \( k \)'th jump of \( Z \) after time \( t \). Let \( n_\eta \) be a random variable \( \mathbb{P} \)-a.s. taking values in \( \mathbb{N} \), with the defining feature that \( T_{n_\eta} = \eta \), so that \( n_\eta \) is the number of jumps after time \( t \) until \( Z \) is absorbed. For notational convenience, introduce

\[
\mathcal{P}^\circ(t) = \frac{1}{\kappa(t)} \mathcal{P}(t)
\]

and \( \beta_s = B^\circ(X^s_{[0,s]}) - B^\circ(X^s_{[0,s-\eta]}) \). We then have on the event \( (\eta < \infty) \):

\[
\mathcal{P}^\circ(t) = \int_{(t,\infty)} \frac{1}{\kappa(s)} B^X_{t-s}X^s_s(ds) + \sum_{t<s<\infty} \beta_s
\]

\[
= \int_{(t,T_1)} \frac{1}{\kappa(s)} B(X^t_{[0,\infty)}, ds) + \beta_{T_1}
\]

\[
+ \sum_{n=1}^{n_\eta-1} \left( \int_{[T_n,T_{n+1})} \frac{1}{\kappa(s)} B(X^T_{T_n,[0,\infty)}, ds) + \beta_{T_{n+1}} \right)
\]

\[
+ \int_{(\eta,\infty)} \frac{1}{\kappa(s)} B(X^\eta_{[0,\infty)}, ds)
\]

by decomposing the integrals between jumps of \( Z \) and using that \( \beta \) is only non-zero at jumps of \( Z \). Hence we can write

\[
\mathcal{P}^\circ(t) = B^\circ(X^t_{[0,T_1)}) - B^\circ(X^t_{[0,t)}) + B^\circ(X^{T_1}_{[0,t)}) - B^\circ(X^t_{[0,T_1)})
\]

\[
+ \sum_{n=1}^{n_\eta-1} \left( B^\circ(X^{T_n}_{T_n,[0,T_{n+1})}) - B^\circ(X^{T_n}_{[0,T_{n+1})}) + B^\circ(X^{T_{n+1}}_{[0,T_{n+1})}) - B^\circ(X^{T_n}_{[0,T_{n+1})}) \right)
\]

\[
+ B^\circ(X^\eta_{[0,\infty)}) - B^\circ(X^\eta_{[0,\eta)})
\]

\[
= B^\circ(X^{T_1}_{[0,T_1)}) - B^\circ(X^t_{[0,t)}) + \sum_{n=1}^{n_\eta-1} \left( B^\circ(X^{T_{n+1}}_{[0,T_{n+1})}) - B^\circ(X^{T_n}_{[0,T_{n})}) \right)
\]

\[
+ B^\circ(X^\eta_{[0,\infty)}) - B^\circ(X^\eta_{[0,\eta)})
\]

Observe that the sum telescopes, so we have

\[
\mathcal{P}^\circ(t) = B^\circ(X^{T_1}_{[0,T_1)}) - B^\circ(X^t_{[0,t)}) + B^\circ(X^\eta_{[0,\eta)}) - B^\circ(X^t_{[0,t)}) + B^\circ(X^\eta_{[0,\infty)}) - B^\circ(X^\eta_{[0,\eta)})
\]

\[
= B^\circ(X^\eta_{[0,\infty)}) - B^\circ(X^t_{[0,t)})
\]

Consequently,

\[
\mathcal{P}(t) = \kappa(t) \mathcal{P}^\circ(t) = \kappa(t) \left( B^\circ(X^\eta_{[0,\infty)}) - B^\circ(X^t_{[0,t)}) \right)
\]

as desired.

\[
\square
\]

**Relation between reserves**

Using Proposition 5.2 and Proposition 5.3, the following theorem is now immediate:
Theorem 5.4. (Representations of transaction time present value.)

For \( t \geq 0 \), it holds that

\[
P(t) = P(t) + \kappa(t) \left( B^o(X^\eta_{[0,t]}) - B^o(X^t_{[0,t]}) \right)
\]

\[
= P(t) + \sum_{t<s<\infty} \kappa(t) \left( B^o(X^s_{[0,t]}) - B^o(X^{s-}_{[0,t]}) \right)
\]

on the event \( (\eta < \infty) \).

Proof. From Propositions 5.2 and 5.3 it follows that on \( (\eta < \infty) \):

\[
P(t) - P(t) = \kappa(t) \left( B^o(X^\eta_{[0,t]}) - B^o(X^t_{[0,t]}) \right),
\]

which implies

\[
P(t) = P(t) + \kappa(t) \left( B^o(X^\eta_{[0,t]}) - B^o(X^t_{[0,t]}) \right).
\]

This proves the first equality. The second equality corresponds to showing that

\[
B^o(X^\eta_{[0,t]}) - B^o(X^t_{[0,t]}) = \sum_{t<s<\infty} \left( B^o(X^s_{[0,t]}) - B^o(X^{s-}_{[0,t]}) \right).
\]

This is trivially satisfied on \( (t \geq \eta) \) since both the left- and right-hand side are zero. Using the same notation as the proof of Proposition 5.3 and the convention \( T_0 = t \), we see on \( (t < \eta) \):

\[
\sum_{t<s<\infty} \left( B^o(X^s_{[0,t]}) - B^o(X^{s-}_{[0,t]}) \right) = \sum_{n=1}^{n_\eta} \left( B^o(X^{T_n}_{[0,t]}) - B^o(X^{T_n-1}_{[0,t]}) \right)
\]

\[
= \sum_{n=1}^{n_\eta} \left( B^o(X^{T_n}_{[0,t]}) - B^o(X^{T_{n-1}}_{[0,t]}) \right)
\]

\[
= B^o(X^\eta_{[0,t]}) - B^o(X^t_{[0,t]})
\]

since the sum telescopes. This establishes the second equality and thus completes the proof.

Remark 5.5. (Relation between reserves in valid and transaction time.)

By taking the conditional expectation given \( \mathcal{F}^\eta_t \) of the expression from Theorem 5.4 we conclude that the expected present value in transaction time \( V \) is different from the classic expected present value in valid time \( V \) in two fundamental ways:

1. It reserves additionally to previously wrongly settled payments, so it is no longer strictly prospective in valid time, in the sense that payments may relate to valid time events that lie before the current point in time.

2. It conditions on the filtration \( \mathcal{F}^\eta \), which is observable, compared to \( \mathcal{F}^X \), which is only partially observable.

Even though the relation between the present values is relatively simple, this does not translate into a simple relation between \( V \) and \( V \) in the general case. This is because we have so far imposed...
very little structure on the model for $Z$, so how the conditional distribution of $Z_s$ given $\mathcal{F}^Z_t$ for $s \geq t$ depends on $\mathcal{F}^Z_t$ can be almost be arbitrarily complicated.

Note that another important consequence of Theorem 5.4 (or rather Proposition 5.3) is that one does not need the distribution of $Z$ to calculate $\mathcal{V}(t)$. The conditional distribution of $X_{[0,\infty)}$ given $\mathcal{F}^Z_t$ is sufficient, since

$$\mathcal{V}(t) = \mathbb{E} \left[ \kappa(t) B^\circ(X_{[0,\infty)}) \mid \mathcal{F}^Z_t \right] - \kappa(t) B^\circ(X^t_{[0,\infty)}).$$

Further, since $X_{[0,\infty)}$ equals $X^\eta_{[0,\infty)}$ almost surely with respect to $\mathbb{P}$ and $X^\eta_{[0,\infty)}$ is $\mathcal{F}^Z_\infty$-measurable, the distribution of $Z$ determines the distribution of $X_{[0,\infty)}$ given $\mathcal{F}^Z_t$. Consequently, the latter might be the natural modeling object.

\[ \blacksquare \]

Remark 5.6. (Non-monotone information.)
Write $\mathcal{V}(t) = g(X_{[0,\infty)})$ for a measurable function $g$, which exists by the Doob-Dynkin lemma. Continuing the discussion from Remark 5.5, standard practice seems to be to use the individual reserve $V^t(t) = g(X^t_{[0,\infty)})$ at time $t$ and use IBNR and RBNS factors on an aggregate level to correct for the fact that typically $X \neq X^t$. Note that the information that one uses for reserving is then non-monotone, since for $0 \leq s \leq t$, it holds that $X^s_{[0,s]}$ is generally unknown from $X^t_{[0,t]}$ and vice versa. Reserves in the presence of non-monotone information have been studied in Christiansen and Furrer (2021). In this, stochastic Thiele differential equations for prospective reserves are derived subject to information deletions, i.e. non-monotone information. These might be useful for studying the properties of the reserves $V^t$ currently used in practice.

\[ \blacksquare \]

Example 5.7. (Total permanent disability: Reserving.)
We here describe reserving in the transaction time model from Example 4.3 and find explicit expressions for $\mathcal{V}$. For this to be tractable mathematically, we impose additional structure on the model for $Z$. The approach taken here is to simplify the model from Example 4.3 considerably, so that one obtains a tractable model for $Z$ and thereby $\mathcal{V}$. For more elaborate insurance contracts this approach becomes difficult, as a realistic model for $Z$ would lead to the distribution of $Z$ being highly complicated. However, cf. Remark 5.5, we do not actually need a full model for $Z$ to obtain explicit expressions for $\mathcal{V}$, we only need a model for $X_{[0,\infty)}$ given $\mathcal{F}^Z_t$. It is outside the scope of this paper to develop methods that exploit this fact.

We simplify the model from Example 4.3 as follows:

- As a convention, we set $Z^{(2)}_0 = 0$, and $Z^{(2)}$ is required to be constant except when $Z^{(1)}$ first jumps to state 2. Given a jump from state 1 to state 2 at time $T^2 = t$, we draw $Z^{(2)}$ independently of $Z^{(1)}_{[T^2,\infty)}$ from a conditional distribution $\mathbb{P}_{Z^{(2)} \mid T^2} (dz \mid t)$ with support on a subset of $(0, t]$.
- $Z^{(1)}$ is a Markov process.
- The jump of $Z^{(1)}$ from state $j$ to state $k$ has transition rate $\mu_{jk}$.

In this case, the information $\mathcal{F}^Z$ is generated by $Z$, so $H^t_s$ can be disregarded. For ease of reference, we redraw Figure 4.1 below.
For this model to be a possible transaction time model corresponding to the valid time model from Example 3.4, the transition rates $\mu_{jk}$ are constrained to be such that the basic bi-temporal structure assumption (iii) is satisfied.

Reserves. Having specified the model, we now write up $V$. First, note that following Example 4.3, we have

$$P(t) = \int_{(t,\infty)} \kappa(t) \frac{\pi_{1}^{1}(s) 1_{(Z_{s}^{(1)} \in \{1,2,3\})}}{\kappa(s)} ds$$

$$+ \left( \frac{\kappa(s)}{\kappa(Z_{s}^{(2)})} b_{1} 1_{[0,\rho]}(Z_{s}^{(2)}) - \int_{[Z_{s}^{(2)},t]} \frac{\kappa(s)}{\kappa(v)} \pi_{1}^{1}(v) dv \right) d \left( 1_{Z_{s}^{(1)} = 4} \right).$$

Let $p_{jk}(t, s) = P(Z_{s}^{(1)} = k \mid Z_{t}^{(1)} = j)$. On $(Z_{t}^{(1)} \in \{4, 5\})$, it holds that $P(t) = 0$ so also trivially $V(t) = 0$. On $(Z_{t}^{(1)} = j)$ for $j \in \{2, 3\}$, it holds that

$$V(t) = \int_{(t,\infty)} \kappa(t) \frac{\pi_{1}^{1}(s) 1_{[k \in \{1,2,3\}}}{\kappa(s)} p_{11}(t, s) \left( \pi_{1}^{1}(s) 1_{[k \in \{1,2,3\}} + \left( \frac{\kappa(s)}{\kappa(Z_{s}^{(2)})} b_{1} 1_{[0,\rho]}(Z_{s}^{(2)}) - \int_{[Z_{s}^{(2)},t]} \frac{\kappa(s)}{\kappa(v)} \pi_{1}^{1}(v) dv \right) \mu_{24}(s) 1_{[k = 2]} \right) ds.$$

We denote the right-hand side by $V_{j}(t, Z_{t}^{(2)})$. On $(Z_{t}^{(1)} = 1)$, we obtain:

$$V(t) = \int_{(t,\infty)} \kappa(t) \frac{\pi_{1}^{1}(s)}{\kappa(s)} p_{11}(t, s) \left( \pi_{1}^{1}(s) + \int_{[0,s]} V_{2}(s, z) P_{Z^{(2)}|T_{2}}(dz \mid s) \mu_{12}(s) \right) ds,$$

which is comparable to a Thiele’s integral equation of type 1; for the latter concept see for instance Christiansen (2012).

To use these expressions in practice, one would need to estimate the transition rates $\mu_{jk}$ and the conditional distribution $P_{Z^{(2)}|T_{2}}(dz \mid t)$. Developing estimation procedures is outside the scope of this paper, but it constitutes an interesting direction for further research.

Reserves: alternative expressions. We may also derive alternative expressions that are easier to relate to valid time model components as well as the claims reserving literature. In general, we would prefer to work as much as possible with the simpler valid time model, as specifying a realistic model for the transaction time process might be difficult.

Define the jump times for the valid time process $\tau^{j} = \inf\{ s \geq 0 : X_{s} = j \}$ for $j \in \{i, d\}$. We first consider the case with $(Z_{t}^{(1)} = 1)$, corresponding to an insured that has yet to report a
disability. Using the tower property with respect to $\tau^i$ on $(Z_t^{(1)} = 1)$ yields
\[
\mathcal{V}(t) = \int_{[0,\infty]} \mathbb{E}[P(t) \mid Z_t^{(1)} = 1, \tau^i = s] \mathbb{P}(\tau^i \in ds \mid Z_t^{(1)} = 1).
\]

We see that the part $\int_{(0,t]} \mathbb{E}[P(t) \mid Z_t^{(1)} = 1, \tau^i = s] \mathbb{P}(\tau^i \in ds \mid Z_t^{(1)} = 1)$, where we integrate over $[0,t]$, represents the IBNR reserve since it relates to disabilities occurring before time $t$, while $\int_{(t,\infty]} \mathbb{E}[P(t) \mid Z_t^{(1)} = 1, \tau^i = s] \mathbb{P}(\tau^i \in ds \mid Z_t^{(1)} = 1)$, where we integrate over $(t, \infty]$, represents the CBNI (Covered-But-Not-Incurred) reserve since it relates to disabilities occurring in the future or not at all.

For the term with $\tau^i = \infty$, it is useful to note that $1_{(\tau^i=\infty)} = 1_{(Z_t^{(1)}=5)}$ by the basic bi-temporal structure assumptions. Elementary calculations, see also Section 4C of [Norberg (1991)], show that for $t \leq s < \infty$ and $j, k \in \{1, 2, 3, 4, 5\}$, the Markov property of $Z^{(1)}$ implies
\[
\mathbb{P}(Z_s^{(1)} = k \mid Z_t^{(1)} = j, Z_{\infty}^{(1)} = 5) = \frac{p_{jk}(t, s)}{p_{j5}(t, \infty)}
\]
when $p_{j5}(t, \infty) > 0$. Hence assuming that $p_{15}(t, \infty) > 0$, we may write
\[
\mathbb{E}[P(t) \mid Z_t^{(1)} = 1, \tau^i = \infty] \mathbb{P}(\tau^i = \infty \mid Z_t^{(1)} = 1) = \int_{(t,\infty]} \frac{\kappa(t)}{\kappa(s)} \sum_{k=1}^{3} \pi_{1,0}^{(k)}(s) p_{1k}(t, s) \frac{p_{k5}(s, \infty)}{p_{15}(t, \infty)} ds p_{15}(t, \infty)
\]
\[
= \int_{(t,\infty]} \frac{\kappa(t)}{\kappa(s)} \sum_{k=1}^{3} \pi_{1,0}^{(k)}(s) p_{1k}(t, s) p_{k5}(s, \infty) ds.
\]

We note en passant that one for computational reasons could use $\sum_{k=1}^{3} p_{1k}(t, s) p_{k5}(s, \infty) = p_{15}(t, \infty) - p_{15}(t, s) p_{55}(s, \infty)$, which is a consequence of the Chapman–Kolmogorov equations.

For the term with $\tau^i < \infty$, we observe
\[
\mathbb{E}[P(t) \mid Z_t^{(1)} = 1, \tau^i = s] = \int_{(t,s]} \frac{\kappa(t)}{\kappa(v)} \pi_{1,0}^{(1)}(v) dv + \frac{\kappa(t)}{\kappa(s)} b_{1,0}(s)
\]
for $s \geq t$ and
\[
\mathbb{E}[P(t) \mid Z_t^{(1)} = 1, \tau^i = s] = -\int_{(s,t]} \frac{\kappa(t)}{\kappa(v)} \pi_{1,0}^{(1)}(v) dv + \frac{\kappa(t)}{\kappa(s)} b_{1,0}(s)
\]
for $s < t$. Define $p_{jk}^X(t, s) = \mathbb{P}(X_s = k \mid X_t = j)$ for $j, k \in \{a, i, d\}$. Using Bayes’ theorem, we can write
\[
\mathbb{P}(\tau^i \in ds \mid Z_t^{(1)} = 1) = \mathbb{P}(Z_t^{(1)} = 1 \mid \tau^i = s) \frac{\mathbb{P}(\tau^i \in ds)}{\mathbb{P}(Z_t^{(1)} = 1)}
\]
\[
= \mathbb{P}(Z_t^{(1)} = 1 \mid \tau^i = s) \mu_{ai}(s) \frac{p_{ja}^X(0, s)}{p_{j1}(0, t)} ds.
\]
This shows that for the IBNR term, the disability rate $\mu_{ai}$ has to be multiplied by the reporting delay distribution $\mathbb{P}(Z_t^{(1)} = 1 \mid \tau^i = s)$ similarly to the Poisson process model in [Norberg (1999)]. The interpretation is that one has to hold a disability reserve for the expected number of disabilities.
\( \mu_{ai}(s) \) ds at a prior time \( s \) times the proportion \( \mathbb{P}(Z_t^{(1)} = 1 \mid \tau^i = s) \) of insured that have yet to report their claim by time \( t \). The extra factor \( \frac{p_{11}^X(0,s)}{p_{11}(0,t)} \) adjusts for the fact that there can be at most one disability occurrence in this model as opposed to a Poisson process model where there can be several occurrences. Note that for \( t < s \), it holds that \( \mathbb{P}(Z_t^{(1)} = 1 \mid \tau^i = s) = 1 \) since we assumed that \( Z^{(2)} \) given \( T^2 = u \) had support on a subset of \( (0,u] \), so a claim cannot be reported before it occurs. In total, we obtain that on \( (Z_t^{(1)} = 1) \):  

\[
\mathcal{V}(t) = \int_{(0,t]} \left( -\int_{(s,t]} \frac{\kappa(t)}{\kappa(v)} \pi 1_{[0,\rho]}(v) \, dv + \frac{\kappa(t)}{\kappa(s)} b 1_{[0,\rho]}(s) \right) \mathbb{P}(Z_t = 1 \mid \tau^i = s) \mu_{ai}(s) \frac{p_{11}^X(0,s)}{p_{11}(0,t)} \, ds \\
+ \int_{(t,\infty)} \left( \int_{(t,s]} \frac{\kappa(t)}{\kappa(v)} \pi 1_{[0,\rho]}(v) \, dv + \frac{\kappa(t)}{\kappa(s)} b 1_{[0,\rho]}(s) \right) \mu_{ai}(s) \frac{p_{11}^X(0,s)}{p_{11}(0,t)} \, ds \\
+ \int_{(t,\infty)} \frac{\kappa(t)}{\kappa(s)} \sum_{k=1}^{3} \pi 1_{[0,\rho]}(s) p_{ik}(t,s) p_{k5}(s,\infty) \, ds.
\]

As stated before, the first term is the IBNR reserve, while the last two represent the CBNI reserve.

We now consider the second case with \( (Z_t^{(1)} = j) \) for \( j \in \{2,3\} \), corresponding to an insured that has reported a disability and is still alive, but where the claim has not yet been awarded. Using the tower property with respect to \( 1_{(\tau < \infty)} \) and on \( (Z_t^{(1)} = j) \), where \( j \in \{2,3\} \), yields  

\[
\mathcal{V}(t) = \mathbb{P}(\tau^i < \infty \mid \mathcal{F}_t^Z) \mathbb{E} \left[ \mathcal{P}(t) \mid \mathcal{F}_t^Z, \tau^i < \infty \right] \\
+ \mathbb{P}(\tau^i = \infty \mid \mathcal{F}_t^Z) \mathbb{E} \left[ \int_{(t,\infty)} \frac{\kappa(t)}{\kappa(s)} \pi 1_{[0,\rho]}(s) 1_{(Z_s^{(1)} \in \{1,2,3\})} \, ds \, \bigg| \mathcal{F}_t^Z, \tau^i = \infty \right] \\
= \mathbb{P}(\tau^i < \infty \mid \mathcal{F}_t^Z) \left( \frac{\kappa(t)}{\kappa(Z_t^{(2)})} b 1_{[0,\rho]}(Z_t^{(2)}) - \int_{[Z_t^{(2)},t]} \frac{\kappa(t)}{\kappa(v)} \pi 1_{[0,\rho]}(v) \, dv \right) \\
+ \int_{(t,\infty)} \frac{\kappa(t)}{\kappa(s)} \sum_{k=1}^{3} \pi 1_{[0,\rho]}(s) p_{jk}(t,s) p_{k5}(s,\infty) \, ds.
\]

This represents the RBNS reserve. We note that with probability \( \mathbb{P}(\tau^i < \infty \mid \mathcal{F}_t^Z) \), the disability benefit is awarded, which results in a disability benefit \( b \) and backpay of some premiums of rate \( \pi \). If the disability claim is not awarded, we reserve for premiums of rate \( \pi \) until the death of the insured.

The above example serves as a simple theoretical demonstration of the potential of our general framework. It could be relevant to develop estimation procedures for this example. Note that in the model of the example, only one disability claim may be reported by the insurer, but an arbitrary amount of rejections and reapplications can take place. The model of the example may therefore serve as an approximation to a more realistic model, where it is possible for the insured to experience and report several distinct disabilities. This approximation is reasonable if the risk period \( \rho \) is small, for instance one year. It would be of great interest to introduce and explore a more advanced transaction time model capable of capturing a wider range of disability insurance products. While this is outside our scope, the framework readily allows for such continued studies.

**Reserve dynamics**

The study of reserve dynamics is of great importance, especially in relation to model validation, Cantelli’s theorem and reserve-dependent payments, Hattendorff’s theorem on non-correlation.
between losses, and the emergence and decomposition of surplus as well as sensitivity analyses, cf. Section 1 in Christiansen and Furrer (2021). We conclude this section by deriving the dynamics of the transaction time reserve \( \mathcal{V} \) following the same procedure as for the classic reserve, see e.g. Christiansen and Djehiche (2020). Essentially, this amounts to applying an explicit martingale representation theorem to \( \mathcal{V} \); the idea of applying martingale representation techniques dates back to Norberg (1992).

Recall the definitions \( \mathcal{V}(t) = \mathbb{E}[\mathcal{P}(t) \mid \mathcal{F}_t^Z] \) and \( Z_t = (Z_t, H^1_t) \). Define a random counting measure \( \mu_Z \) corresponding to \( Z \) by

\[
\mu_Z(C) = \sum_{n=1}^{\infty} 1_{C}(T_n, Z_{T_n}), \quad C \in \mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbb{R}^\infty),
\]

and let \( \Lambda_Z \) be its compensating measure, given in Definition 4.3.2 (iii) of Jacobsen (2006). By Theorem 4.5.2 of Jacobsen (2006), if \( \mathbb{E}[\mu_Z([0,t] \times D)] < \infty \) for all \( t \geq 0 \) and \( D \in \mathcal{B}(\mathbb{R}^\infty) \), we have that

\[
t \mapsto \mu_Z([0,t] \times D) - \Lambda_Z([0,t] \times D)
\]

is a martingale for any \( D \in \mathcal{B}(\mathbb{R}^\infty) \). Let \( \xi_n = (T_1, ..., T_n; Z_{T_1}, ..., Z_{T_n}) \) be the MPP history of \( Z \) at time \( T_n \).

Write \( y = (z^y, h^y) \) for a generic realization of \( Z \), where the coordinates \( z^y \) and \( h^y \) pertain to \( Z_t \) and \( H^1_t \), respectively. Write \( n_{h^y} \) for the number of jumps in \( h^y \). Finally, define the difference in payments at time \( t \) between a jump-to-\( h^y \) and a remain-in-\( h^w \) scenario:

\[
B_t(h^w, h^y) = B_t^{n_{h^w}^{\text{in}}(t-), n_{h^y}^{\text{out}}(t)}(\{t\}) + \kappa(t)\left( B_t^\varnothing((f_{h^w}^{n_{h^w}^{\text{in}}}(s))_{0 \leq s < t}) - B_t^\varnothing((f_{h^y}^{n_{h^y}^{\text{in}}}(s))_{0 \leq s < t}) \right)
- B_t^{n_{h^w}^{\text{in}}(t-), n_{h^y}^{\text{out}}(t)}(\{t\}),
\]

and the sums at risk in the transaction time model for a jump of \( Z \) to \( y \) at time \( t \):

\[
\mathcal{R}(t, y) = \sum_{n=1}^{\infty} 1_{(T_n < t \leq T_{n+1})}(B_t(H_{T_{n+1}}^+, h^y)) + \mathbb{E}[\mathcal{P}(t) \mid \xi_n, (T_{n+1}, Z_{T_{n+1}}) = (t, y)]
- \mathbb{E}[\mathcal{P}(t) \mid \xi_n, T_{n+1} > t].
\]

The expression \( \mathbb{E}[\mathcal{P}(t) \mid \xi_n, T_{n+1} > t] \) is understood as the conditional expectation of \( \mathcal{P}(t) \) given \( \xi_n \) and \( 1_{(T_{n+1} > t)} \) considered only on the set \( (T_{n+1} > t) \). We then have the following theorem:

**Theorem 5.8. (Transaction time reserve dynamics.)**

For \( t \geq 0 \), it holds that

\[
\mathcal{V}(dt) = \mathcal{V}(t^-) - \frac{\kappa(dt)}{\kappa(t^-)} - \mathcal{B}(dt) + \int_{\mathbb{R}^\infty} \mathcal{R}(t, y) (\mu_Z - \Lambda_Z)(dt, dy).
\]  

**Proof.** Introduce

\[
\mathcal{P}^\circ(t) = \frac{1}{\kappa(t)} \mathcal{P}(t).
\]

We have that \( \mathcal{P}^\circ(0) = \mathcal{P}(0) \), which is assumed integrable, so we can define

\[
t \mapsto M_t = \mathbb{E}[\mathcal{P}^\circ(0) \mid \mathcal{F}_t^Z],
\]

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which is a martingale. Since $V$ is presumed $\mathcal{F}^Z$-adapted and $\mathbb{P}$-a.s. right-continuous, the same holds for a version of $M$, since $M_t = \frac{1}{\kappa(t)} \mathcal{Z}(t) + \mathcal{P}_0(0) - \mathcal{P}(t)$. Then a martingale representation theorem, namely Theorem 4.6.1 of Jacobsen (2006), gives the existence of predictable processes $S^n_t$ such that

$$M_t = M_0 + \int_{(0,t] \times \mathbb{R}^n} S^n_t (\mu_Z - \Lambda_Z)(ds, dy)$$

$\mathbb{P}$-a.s. simultaneously over $t$. Using the adaptedness of $M$, we can, as in the proof of the aforementioned Theorem 4.6.1, write

$$M_t = \sum_{n=0}^{\infty} 1_{(T_n < t \leq T_{n+1})} g^n_{\xi_n}(t)$$

for measurable functions $(h_n, t) \mapsto g^n_{h_n}(t)$. Due to $M$ being a conditional expectation, we can use Corollary 4.2.2 of Jacobsen (2006) to identify

$$g^n_{\xi_n}(t) = \mathbb{E}[\mathcal{P}_0(0) \mid \xi_n, T_{n+1} > t].$$

The proof of the aforementioned Theorem 4.6.1 furthermore gives that

$$S^n_t = \sum_{n=0}^{\infty} 1_{(T_n < t \leq T_{n+1})} \left(g^n_{\xi_n(t,y)}(t) - g^n_{\xi_n}(t)\right)$$

such that

$$S^n_t = \sum_{n=0}^{\infty} 1_{(T_n < t \leq T_{n+1})} \left(\mathbb{E}[\mathcal{P}_0(0) \mid \xi_n, (T_{n+1}, \mathcal{Z}_{T_{n+1}}) = (t, y)] - \mathbb{E}[\mathcal{P}_0(0) \mid \xi_n, T_{n+1} > t]\right)$$

$$= \sum_{n=0}^{\infty} 1_{(T_n < t \leq T_{n+1})} \left(\mathbb{E}[\mathcal{P}_0(t-) \mid \xi_n, (T_{n+1}, \mathcal{Z}_{T_{n+1}}) = (t, y)] - \mathbb{E}[\mathcal{P}_0(t-) \mid \xi_n, T_{n+1} > t]\right)$$

using that $\mathcal{P}_0(0) - \mathcal{P}_0(t-) = \int_{(0,t]} \frac{1}{\kappa(s)} B(ds)$ are $\xi_n$-measurable on $(T_n < t \leq T_{n+1})$. Therefore, the dynamics of $M$ is

$$dM_t = \int_{\mathbb{R}^n} S^n_t (\mu_Z - \Lambda_Z)(dt, dy)$$

$$= \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} 1_{(T_n < t \leq T_{n+1})} \left(\mathbb{E}[\mathcal{P}_0(t-) \mid \xi_n, (T_{n+1}, \mathcal{Z}_{T_{n+1}}) = (t, y)] - \mathbb{E}[\mathcal{P}_0(t-) \mid \xi_n, T_{n+1} > t]\right) (\mu_Z - \Lambda_Z)(dt, dy).$$

Using that

$$\mathcal{P}_0(t) - \mathcal{P}_0(0) = - \int_{(0,t]} \frac{1}{\kappa(s)} B(ds)$$

is $\mathcal{F}_t^Z$ adapted, we get

$$\mathbb{E}[\mathcal{P}_0(0) \mid \mathcal{F}_t^Z] - \mathbb{E}[\mathcal{P}_0(0) \mid \mathcal{F}_0^Z] = \mathbb{E}[\mathcal{P}_0(t) \mid \mathcal{F}_t^Z] - \mathbb{E}[\mathcal{P}_0(0) \mid \mathcal{F}_0^Z] - (\mathcal{P}_0(t) - \mathcal{P}_0(0)),$$

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which upon rearrangement becomes

\[ \mathbb{E}[\mathcal{P}^o(t) | \mathcal{F}_t^Z] - \mathbb{E}[\mathcal{P}^o(0) | \mathcal{F}_0^Z] = \mathcal{P}^o(t) - \mathcal{P}^o(0) + \mathbb{E}[\mathcal{P}^o(0) | \mathcal{F}_t^Z] - \mathbb{E}[\mathcal{P}^o(0) | \mathcal{F}_0^Z]. \]

Introducing

\[ \mathcal{V}^o(t) = \frac{1}{\kappa(t)} \mathcal{V}(t) = \mathbb{E}[\mathcal{P}^o(t) | \mathcal{F}_t^Z], \]

we can write this as

\[ \mathcal{V}^o(t) - \mathcal{V}^o(0) = \mathcal{P}^o(t) - \mathcal{P}^o(0) + M(t) - M(0). \]

The identity \[ 5.4 \] furthermore gives

\[ \mathcal{P}^o(dt) = -\frac{1}{\kappa(t)} \mathcal{B}(dt). \]

The above calculations imply

\[ \mathcal{V}^o(dt) = \mathcal{P}^o(dt) + dM_t = -\frac{1}{\kappa(t)} \mathcal{B}(dt) + \sum_{n=0}^{\infty} \int_{\mathbb{R}^\infty} 1_{(T_n < t \leq T_{n+1})} \left( \mathbb{E}[\mathcal{P}^o(t-) | \xi_n, (T_{n+1}, \mathcal{Z}_{T_{n+1}}) = (t, y)] - \mathbb{E}[\mathcal{P}^o(t-) | \xi_n, T_{n+1} > t] \right) (\mu_Z - \Lambda_Z)(dt, dy). \]

The time \( t \) payment \( \mathcal{B}(\{t\}) \) can be taken out of both intergrands, and this amounts to \( \mathcal{B}_t(H_{t_{\xi_n}}^{T_n}, y^\nu) \). It is the difference in the payment at time \( t \) between a jump and a remain scenario when \( Z \) jumps to \( y \). Taking out the time \( t \) payment \( \mathcal{P}^o(t-) = \frac{1}{\kappa(t)} \mathcal{B}(\{t\}) + \mathcal{P}^o(t) \) and using integration by parts, we finally have

\[ \mathcal{V}(dt) = d(\kappa(t)) \mathcal{V}(t) = \mathcal{V}^o(t-) \kappa(dt) + \kappa(t) \mathcal{V}^o(dt) = \mathcal{V}(t-) \frac{\kappa(dt)}{\kappa(t)} - \mathcal{B}(dt) + \sum_{n=1}^{\infty} \int_{\mathbb{R}^\infty} 1_{(T_n < t \leq T_{n+1})} \left( \mathcal{B}_t(H_{t_{\xi_n}}^{T_n}, y^\nu) + \mathbb{E}[\mathcal{P}(t) | \xi_n, (T_{n+1}, \mathcal{Z}_{T_{n+1}}) = (t, y)] - \mathbb{E}[\mathcal{P}(t) | \xi_n, T_{n+1} > t] \right) (\mu_Z - \Lambda_Z)(dt, dy), \]

which yields the desired result by definition of the sums at risk.

Theorem \[ 5.8 \] shows that the transaction time reserve \( \mathcal{V} \) changes with interest accrual \( \mathcal{V}(t-) \frac{\kappa(dt)}{\kappa(t-)} \), actual benefits less premiums \( \mathcal{B}(dt) \) and a martingale term \( \int_{\mathbb{R}^\infty} \mathcal{R}(t, y) (\mu_Z - \Lambda_Z)(dt, dy) \), which is the sums at risk integrated with respect to the underlying compensated random counting measure. The martingale term may be interpreted as stochastic noise since it is a mean-zero process, and may thus be used for model validation and back-testing purposes. Actual applications are outside the scope of this paper.

One could alternatively have derived Theorem \[ 5.8 \] from Theorem 7.1 in Christiansen (2021), which is an explicit martingale representation theorem that holds even when the information being conditioned on is non-monotone. The proof presented here is however more concise, as our information \( \mathcal{F}^Z \) is monotone, so more standard results apply. Theorem \[ 5.8 \] is similar to Proposition 3.2 in Christiansen and Djehiche (2020), but differs among other things by not being restricted to state processes taking values in a finite space.
Remark 5.9. (Dynamics of valid time reserve.)

Define the random counting measure $\mu_X$ corresponding to $X$ by

$$\mu_X(C) = \sum_{n=1}^{\infty} 1_C(\tau_n, X_{\tau_n}), \quad C \in \mathcal{B}([0, \infty)) \otimes \mathbb{B}(\mathbb{R}^d).$$

Let $\Lambda_X$ be the compensating measure for $\mu_X$, and let $\zeta_n = (\tau_1, \ldots, \tau_n; X_{\tau_1}, \ldots, X_{\tau_n})$ be the MPP history of $X$ at time $\tau_n$. By the same calculations as for Theorem 5.8 we find the dynamics of the valid time reserve $V$:

$$V(dt) = V(t-) \frac{\kappa(dt)}{\kappa(t-)} - B(dt) + \int_{\mathbb{R}^d} R(t, y) (\mu_X - \Lambda_X)(dt, dy) \quad (5.5)$$

for the sums at risk

$$R(t, y) = \sum_{n=1}^{\infty} 1_{(\tau_n < t \leq \tau_{n+1})} \left( B((X_{[0,t]}, y), \{t\}) - B((X_{[0,t]}, X_{t-}), \{t\}) + \mathbb{E}[P(t) \mid \zeta_n, (\tau_{n+1}, X_{\tau_{n+1}}) = (t, y)] - \mathbb{E}[P(t) \mid \zeta_n, \tau_{n+1} > t] \right),$$

where $(X_{[0,t]}, x)$ is notation for the process that is $X$ on $[0, t)$ and $x$ on $\{t\}$. This result is again similar to Proposition 3.2 in Christiansen and Djehiche [2020], but still differs among other things by not being restricted to state processes taking values in a finite space.

Suppose now that $X$ is a pure Markov jump process on a finite state space $E = \{1, 2, \ldots, J\}$ with payments specified as in Example 3.2. In other words, the valid time payments consist of deterministic sojourn payments $t \mapsto B_j(t)$ and deterministic transition payments $t \mapsto b_{jk}(t)$. Then

$$B((X_{[0,t]}, y), \{t\}) - B((X_{[0,t]}, X_{t-}), \{t\}) = b_{X_{t-}y}(t).$$

Furthermore,

$$\mu_X(dt, \{k\}) = N_{X_{t-k}}(dt)$$

and, since $X$ is Markovian,

$$\Lambda_X(dt, \{k\}) = \Lambda_{X_{t-k}}(dt)$$

for suitably regular cumulative transition rates $t \mapsto \Lambda_{jk}(t)$. Consequently, by invoking the Markov property, the dynamics (5.5) read

$$V(dt) = V(t-) \frac{\kappa(dt)}{\kappa(t-)} - B(dt)$$

$$+ \sum_{j,k=1}^{J} 1_{(X_{t-}=j)}(b_{jk}(t) + \mathbb{E}[P(t) \mid X_t = k] - \mathbb{E}[P(t) \mid X_t = j])(N_{jk}(dt) - \Lambda_{jk}(dt)) \quad (5.6)$$

This constitutes a significant simplification. □

In comparing (5.3) with (5.5), it is apparent that the transaction and valid time reserves admit comparable dynamics. In both cases, there is a contribution due to interest accrual, a contribution from benefits less premiums, and finally a martingale term. In general, the dynamics of the
transaction time reserve are more complicated than that of the valid time reserve – for two reasons. First, the martingale term is more involved, which stems from the fact that the model for \( Z \) is typically more elaborate than that for \( X \). Second, the accumulated cash flow in transaction time \( B \) is a complicated function of, among other things, the accumulated cash flow in valid time \( B \). The difference might be particularly striking under the quite common assumption that \( X \) is a pure Markov jump process on a finite state space \( E = \{1, 2, \ldots, J\} \) and the valid time payments consist of deterministic sojourn and transition payments. In this case, the dynamics of the valid time reserve simplify, cf. (5.6), but there is in general no reason why this simplification should carry over to the transaction time reserve – unless further assumptions are imposed.

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References


