# Cash flows and policyholder behaviour in the semi-Markov life insurance setup

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Abstract

Within the setup of a semi-Markov process in a finite state space, we consider a life insurance contract. First, without the modelling of policyholder behaviour, we show how to calculate the expected cash flow associated with future payments, and to that end we present a version of Kolmogorov's forward integro-differential equation. The semi-Markov model is then extended to include modelling of surrender and free policy behaviour, and the main result is a modification of Kolmogorov's foward integro-differential equation, such that the cash flow can be calculated without significantly more complexity than the cash flow without policyholder modelling. The result is also demonstrated for the traditional Markov case where there is no duration dependence, and numerical examples are studied.

Keywords: duration dependence, Kolmogorov's differential equations, surrender, free policy

## 1 Introduction

In this paper we consider the problem of valuation of prospective reserves and expected cash flows for life insurance liabilities, including the modelling of policyholder behaviour. The setup consists of a semi-Markov process for the state of the insured in a multi-state model, and the special case of a Markov process is considered as well. When we include policyholder behaviour in the model, in particular the so-called free policy option[2], an

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[2]"Free policy" is sometimes referred to as "paid-up policy" in the literature.

extra duration dependence arises, such that we have a double duration setup. The main result of the paper is that we can effectively eliminate the extra duration. Thus, we simply have to solve a modified Kolmogorov forward integro-differential equation, which in complexity is equal to the differential equation without policyholder behaviour.

The setup of a semi-Markov model allows for dependence on the duration in the current state. Thus, the transition rates can be dependent on the duration, and the payment functions may also be dependent on the duration. An example of the presence of duration dependence in the transition rates is within a disability model, where a 50 year old disabled person might have a higher recovery transition rate if disability occured half a year ago than if disability occured 10 years ago, and hence the transition rate is not only age-dependent. For empirical evidence, see [22] and [10], wherein the recovery rate and the mortality rate for disabled are shown to be dependent on time since disability as well as age. An example of duration dependence in the payment functions is a disability annuity where one can model a 3 month waiting period after disability before the annuity starts. If only the payment functions depend on the duration, the semi-Markov process simplifies to a Markov process, however, in order to valuate the prospective reserve and/or the expected cash flow, one needs the duration dependent transition probabilities, and thus the duration dependent formulae are needed.

In terms of understanding the characteristics of the life insurance liabilities, in particular the future cash flow, it is of importance to take policyholder behaviour into account. This aspect is also given considerable attention from a Solvency 2 perspective, where life insurers are required to take into account policyholder behaviour when valuating the liabilities, see Section 3.5 in [5]. One aim of this paper is to show how the semi-Markov setup can be used to model policyholder behaviour, and how to efficiently calculate cash flows.

In the first part of the paper, Sections 2 and 3, we give an overview of current results for the multi-state semi-Markov life insurance setup, including a version of Kolmogorov's backward and forward differential equations, which does not seem to be well known in the semi-Markov setup. Semi-Markov models in life insurance were first introduced in [14], and later treatments include [19] and [6]. For a more theoretical treatment of the topic, see also [11], who first introduced the approach based on cumulative transition rates for the multi-state semi-Markov life insurance setup. We define the deterministic cash flow associated with the random future payments as the expectation of the future payments, and show how to efficiently valuate it using Kolmogorov's forward integro-differential equations. We specialise to the Markov case where there is no duration dependence, and recover the classic results without duration dependence, which can be found in e.g. [21] or [16].

In the second part of the paper, Section 4, the semi-Markov model is extended by including policyholder behaviour in the form of a surrender and a free policy option. These are modelled by specifying transitions in an extended state space of the semi-Markov model, corresponding to exercises of the options. The surrender option is a right of the policyholder to cancel the contract, and receive an account value calculated on a technical basis. The free policy option is a right of the policyholder to cancel future payments and let the contract continue as a policy with no premiums and with reduced benefits, where the reduction is calculated on a technical basis. An exercise of any of these options thus changes future payments. While the modelling of these options has an effect on the prospective reserve, it is of even greater importance when considering the structure of future cash flows, and the interest rate sensitivity. In this paper, the exercise of the options occurs randomly, and a similar approach is taken in [12] where the insurance risk and policyholder behaviour is modelled by two separate Markov chains, possibly dependent. Earlier studies of random policyholder behaviour modelling include [17] and [18]. In contrast to this is the approach where the surrender occurs rationally, which is studied in [23]. For a comparison and overview of these approaches, see [20]. Attempts to couple rational and random behaviour are done in [9] and [4], where the common approach is that the level of random exercises is dependent on certain rational indicators.

In the extended semi-Markov model, the free policy option introduces a dependence in the payment functions on the duration since the free policy transition. Thus the transition probabilities needed for valuation are dependent on two durations, and we say that we have a setup with a double duration dependence. We show how the dependence on the duration since the free policy transition can be effectively eliminated, which is the main result of this paper. With this result, the evaluation of the prospective reserve and the cash flow can be carried out using a modified version of Kolmogorov's forward integro-differential equation, corresponding to the computational complexity of the original semi-Markov model with one duration. For a portfolio of life insurance contracts, this reduction, measured in time usage, is considerable.

We also present the results for the Markov case, which is the special case of the semi-Markov setup where there is no dependence on the duration. The extension to include policyholder behaviour is shown, and like in the semi-Markov setup, the free policy modelling introduces a dependence on the duration since the free policy transition. Thus, the Markov process becomes a semi-Markov process when the free policy option is included. As in the semi-Markov case, our result effectively eliminates this extra dependence on the duration, and a modification of Kolmogorov's forward differential equation makes it computationally simple to calculate cash flows, effectively as in a Markov model.

We conclude the paper by giving numerical results illustrating the cash flows for a simple life insurance policy. We show how the cash flows change due to policyholder modelling, and highlight that the dollar duration changes significantly, even though the prospective reserve might not change a lot. This is of particular interest if the cash flows are used for the hedging of interest rate risk, for example by duration-matching.

# 2 Setup

We consider the classic life insurance setup with a stochastic process $\mathbf{Z}$ in a finite state space $\mathcal{J}$, denoting the state of the insured, see e.g. [13] and [21], where the process $\mathbf{Z}$ is a Markov chain. In this paper, we let $\mathbf{Z}$ be a semi-Markov process as in [6] and [19]. To each state and each transition between states of $\mathbf{Z}$, we attach payments.

## 2.1 The semi-Markov model

We consider a semi-Markov process $\mathbf{Z} = (Z(t))_{t\geq 0}$ in a state space $\mathcal{J} = \{0, 1, \ldots, J-1\}$ with $J$ states, and let $Z(0) = 0$. Let $\mathbf{U} = (U(t))_{t\geq 0}$ be defined as the duration,

$$U(t) = \sup\{\, s \in [0,t] |\, Z(\tau) = Z(t), \tau \in [t-s, t]\},$$

which implies that $U(t)$ is the time $Z(t)$ has spent in its current state. We assume that $(\mathbf{Z}, \mathbf{U})$ is a Markov process, thus $\mathbf{Z}$ is a semi-Markov process. The processes are defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with the filtration

$$\mathcal{F}(t) = \sigma(\, Z(s), U(s) |\, s \leq t) = \sigma(\, Z(s) |\, s \leq t),$$

with equality because $\mathbf{U}$ is constructed from $\mathbf{Z}$.

We define the transition probabilities for going from state $i$ at time $s$ with duration $u$ to state $j$ at time $t$ with duration less than $z$,

$$p_{ij}\,(s, t, u, z) = P\,(\, Z(t) = j, U(t) \leq z |\, Z(s) = i, U(s) = u)\,, \tag{2.1}$$

and, for $F \subset \mathcal{J}$ and $G \in \mathcal{B}(\mathbb{R})$, we let

$$p_{iF}\,(s, t, u, G) = \sum_{j\in F} \int_G p_{ij}(s, t, u,\, \mathrm{d}z)$$

denote the probability measure of the distribution of $(Z(t), U(t))$ given $Z(s) = i$ and $U(s) = u$. This measure has finite support: With initial duration $u$ at time $s$, the maximum duration at time $t$ is $u + t - s$. Thus, the support of the measure is a subset of $\mathcal{J} \times [0, u+t-s]$, and we frequently use $(0, u+t-s]$ as integration bounds.

The transition rates are, for $i, j \in \mathcal{J}, i \neq j$ and $t, u \geq 0$, defined as

$$\begin{aligned} \mu_{ij}(t,u) &= \lim_{h \searrow 0} \frac{1}{h} p_{ij}\left(t, t+h, u, \infty\right), \\ \mu_{i\cdot}(t,u) &= \sum_{\substack{j \in \mathcal{J} \\ j \neq i}} \mu_{ij}(t,u), \end{aligned} \tag{2.2}$$

and throughout this paper we assume that they exist, and that they are continuous in $t$ and $u$. We see later, in Proposition 2.4, that the transition rates determine the distribution of $\mathbf{Z}$.

Define the counting processes[3]

$$N_{ij}(t) = \#\left\{ s \in [0,t] \,|\, Z(s) = j, Z(s-) = i \right\}.$$

Then $N_{ij}(t)$ counts the number of jumps from state $i$ to state $j$ in the interval $[0,t]$. The counting process $N_{ij}(t)$ has intensity $1_{\{Z(t-)=i\}}\mu_{ij}(t, U(t-))$, and the process $\mathbf{M}_{ij}$ defined by

$$M_{ij}(t) = N_{ij}(t) - \int_0^t 1_{\{Z(s-)=i\}}\mu_{ij}(s, U(s-))\,\mathrm{d}s,$$

is a martingale. Using this, then for any predictable process $H(t)$, we obtain

$$\begin{aligned} \mathrm{E}\left[\int_0^t H(s)\,\mathrm{d}N_{ij}(s)\right] &= \mathrm{E}\left[\int_0^t H(s) 1_{\{Z(s-)=i\}}\mu_{ij}(s, U(s-))\,\mathrm{d}s\right] \\ &= \int_0^t H(s) \int_0^s p_{0i}\left(0, s, 0, \,\mathrm{d}z\right)\mu_{ij}(s,z)\,\mathrm{d}s. \end{aligned} \tag{2.3}$$

For more details on results of this type, see e.g. [3]. Throughout the paper we use the convention $\int_a^b = \int_{(a,b]}$ and $\int_a^\infty = \int_{(a,\infty)}$, such that the left endpoint is excluded from the integration range.

## 2.2 Life insurance payments

We model the payments associated with a life insurance contract by letting the semi-Markov process $\mathbf{Z}$ denote the state of the insured. To each state $i$, we associate a continuous payment rate $b_i(t,u)$ and single payments $\Delta B_i(t,u)$ at time $t$ with duration $u$. Also, for each transition from $i$ to $j$, we associate a single payment $b_{ij}(t,u)$ at time $t$ when the duration in the state $i$ was $u$. We write the total payments at time $t$ as $B(t)$,

[3] The notation $f(x-) = \lim_{y \nearrow x} f(y)$ is used.

and the payment process $\mathbf{B}$ then satisfies

$$\begin{aligned} \mathrm{d}B(t) &= \sum_{i\in\mathcal{J}} 1_{\{Z(t)=i\}}\,\mathrm{d}B_i(t,U(t)) + \sum_{\substack{i,j\in\mathcal{J}\\ i\neq j}} b_{ij}(t,U(t-))\,\mathrm{d}N_{ij}(t),\\ \mathrm{d}B_i(t,U(t)) &= b_i(t,U(t))\,\mathrm{d}t + \Delta B_i(t,U(t)), \end{aligned} \tag{2.4}$$

where $\mathrm{d}B_i(t,u)$ are the payments in state $i$ at time $t$ if the duration is $u$. The functions $b_i$, $\Delta B_i$ and $b_{ij}$ are assumed to be deterministic and piecewise continuous.

## 2.3 Interest rate

Let $r(t)$ denote the continuously compounded interest rate. We assume that $r(t)$ is deterministic, however, the results of this paper can easily be obtained with a stochastic short rate instead, if it is independent of $\mathbf{Z}$. In particular, the main results are about the semi-Markov chain $\mathbf{Z}$ and transition probabilities, and are independent of the interest rate.

## 2.4 Cash flows and valuation

For the balance sheet, one calculates the expected present value of future payments with appropriate interest rate assumptions. This is denoted the prospective reserve.

**Definition 2.1.** *The prospective reserve, $V_{i,u}(t)$, at time $t$ of the future payments of $\mathbf{B}$, given that we are in state $i$ with duration $u$, is*

$$V_{i,u}(t) = \mathrm{E}\left[\int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}B(s)\,\middle|\, Z(t)=i, U(t)=u\right].$$

Using calculations similar to (2.3) and the linearity of conditional expectations yield the well known result

$$\begin{aligned} &V_{i,u}(t)\\ &= \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} \sum_{j\in\mathcal{J}} \int_0^{u+s-t} p_{ij}\left(t,s,u,\mathrm{d}z\right)\left(\mathrm{d}B_j(s,z) + \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \mu_{jk}(s,z) b_{jk}(s,z)\,\mathrm{d}s\right), \end{aligned} \tag{2.5}$$

for more details, see e.g. [14].

The payment at a future time $s$ can formally be written as $\mathrm{d}B(s)$, and we say that $\mathrm{d}B(s)$ is the stochastic cash flow at time $s$. Since it is stochastic, it is not very convenient in practice, and one can be interested in a deterministic version. This can be obtained by taking the expectation of $\mathbf{B}$, thus considering the expected cash flow. For example, if one has a lot of identical and independent life insurance contracts, the total payments are, by the law of large numbers, close to the expected payments. We define the expected cash flow, and simply refer to it as the cash flow.

**Definition 2.2.** *The* cash flow *of a payment process* $\mathbf{B}$ *conditional on* $(Z(t) = i, U(t) = u)$ *is the payments of the deterministic payment process* $(A_{i,u}(t,s))_{s\geq t}$*, defined by*

$$A_{i,u}(t,s) = \mathrm{E}\left[B(s) - B(t)\middle|\, Z(t) = i, U(t) = u\right].$$

Using the cash flow process, the expected payments at a future time $s$ given $(Z(t) = i, U(t) = u)$ can be written as $\mathrm{d}A_{i,u}(t,s) = A_{i,u}(t, \mathrm{d}s)$. The cash flows can be used to find the prospective reserve.

**Proposition 2.3.** *The cash flow* $(A_{i,u}(t,s))_{s\geq t}$ *satisfies*

$$V_{i,u}(t) = \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A_{i,u}(t,s),$$

$$\mathrm{d}A_{i,u}(t,s) = \sum_{j\in\mathcal{J}} \int_0^{u+s-t} p_{ij}\left(t,s,u,\,\mathrm{d}z\right)\left(\mathrm{d}B_j(s,z) + \sum_{\substack{k\in\mathcal{J}\\k\neq j}} \mu_{jk}(s,z) b_{jk}(s,z)\,\mathrm{d}s\right).$$

*Proof.* Let $t \geq 0$. Define $\widetilde{B}_t(s) = B(s) - B(t)$ and note that Definition 2.1 can be written as

$$V_{i,u}(t) = \mathrm{E}\left[\int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\widetilde{B}_t(s)\middle|\, Z(t) = i, U(t) = u\right].$$

Let $F_t(s) = e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}$, and use integration by parts to obtain,

$$\int_t^\infty F_t(s)\,\mathrm{d}\widetilde{B}_t(s) = F_t(\infty)\widetilde{B}_t(\infty) - F_t(t)\widetilde{B}_t(t) - \int_t^\infty \widetilde{B}_t(s)\,\mathrm{d}F_t(s).$$

Applying conditional expectation on both sides, conditioning on $(Z(t) = i, U(t) = u)$, yields

$$V_{i,u}(t) = F_t(\infty)A_{i,u}(t,\infty) - F_t(t)A_{i,u}(t,t) - \int_t^\infty A_{i,u}(t,s)\,\mathrm{d}F_t(s),$$

and the first result follows by integration by parts on the right hand side. The second result follows immediately from the first result and equation (2.5). ☐

## 2.5 Kolmogorov's backward differential equation

Kolmogorov's backward differential equation provides a way to calculate the transition probabilities given that the transition rates are specified. Here, we present a version for the semi-Markov chain, which can be found as a partial differential equation in [6] and in [11]. We present the differential equation as an ordinary differential equation. The differential equation can also be obtained from a similar integral equation version found in [14], by differentiation and some further calculations.

**Proposition 2.4.** (Kolmogorov's backward differential equation) *Let* $0 \leq t_0 \leq t$ *and* $d, u \geq 0$ *and* $j \in \mathcal{J}$*. Define* $D(s) = d + s - t_0$*. The transition probabilities* $p_{ij}(t_0, t, d, u)$*, for* $i \in \mathcal{J}$*, satisfy the system of differential equations given by,*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(s, t, D(s), u) &= \mu_{i\cdot}(s, D(s)) p_{ij}(s, t, D(s), u) \\ &\quad - \sum_{\substack{k \in \mathcal{J} \\ k \neq i}} \mu_{ik}(s, D(s)) p_{kj}(s, t, 0, u), \\ p_{ij}(t, t, D(t), u) &= 1_{\{i=j\}} 1_{\{D(t) \leq u\}}. \end{aligned} \tag{2.6}$$

From Kolmogorov's backward differential equations, we see in particular that the transition rates determine the distribution of $Z(t)$.

Solving Kolmogorov's backward differential equation, (2.6), is done for fixed $j \in \mathcal{J}$, $t_0 < t$ and $d, u \geq 0$ and yields the probabilities $(p_{ij}(t_0, t, d, u))_{i \in \mathcal{J}}$. We solve the system of differential equations (2.6), starting with boundary conditions $p_{ij}(t, t, d + t - t_0, u)$, $i \in \mathcal{J}$ and solving backwards to time $t_0$, corresponding to the blue line in Figure 1. This is not straightforward, since the quantities $(p_{ij}(s, t, 0, u))_{i \in \mathcal{J}}$ for $s \in (t_0, t)$ are needed, corresponding to the values on the red line in Figure 1. Thus, these values need to be calculated first. For fixed $j \in \mathcal{J}$, $t_0 \leq t$ and $d, u \geq 0$, we outline an algorithm for calculating the probabilities $(p_{ij}(t_0, t, d, u))_{i \in \mathcal{J}}$, with a numerical procedure using step size $h$, corresponding to a grid with step size $h$. We assume that $t_0, d$ and $t, u$ lie on the grid, in particular that there is $N \in \mathbb{N}$ such that $t = t_0 + Nh$.

1. With boundary condition $p_{ij}(t, t, h, u)$, calculate $p_{ij}(t - h, t, 0, u)$ for each $i \in \mathcal{J}$. Note, that the numerical solution is calculated in one step, so no values $(p_{ij}(s, t, 0, u))_{i \in \mathcal{J}}$ are needed for $s \in (t - h, t)$.
2. With boundary condition $p_{ij}(t, t, 2h, u)$, calculate $p_{ij}(t - 2h, t, 0, u)$. Here, the numerical algorithm requires two steps, where the values $(p_{ij}(t - h, t, 0, u))_{i \in \mathcal{J}}$ are used in the middle step.

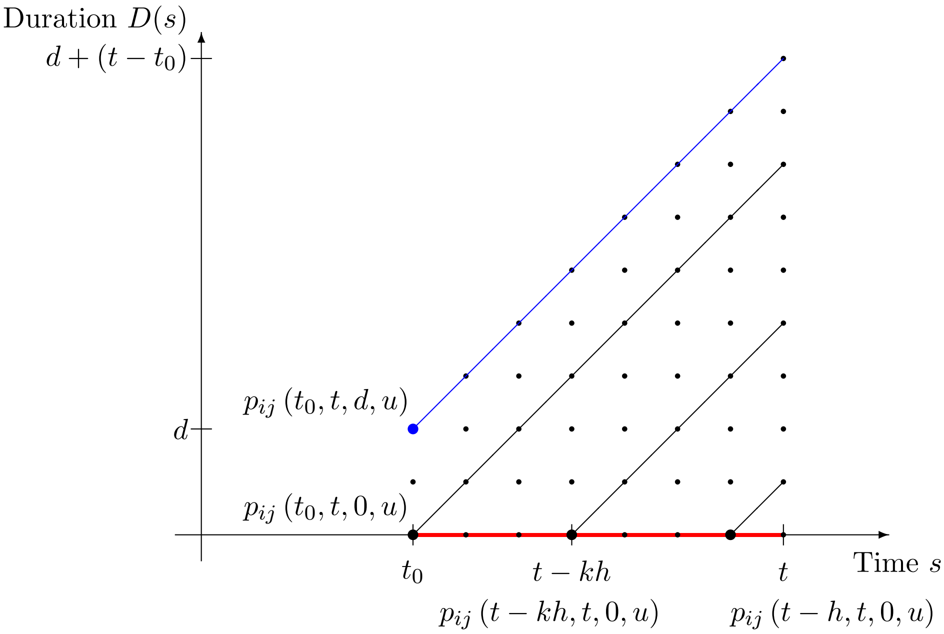


**Figure 1:** *Sketch of algorithm to calculate the transition probabilities $p_{ij}(t_0, t, d, u)$ for all $i \in \mathcal{J}$ using Kolmogorov's backward differential equation, using a numerical algorithm with step size $h$, matching the drawn grid. First, for all values $p_{ij}(s, t, 0, u)$, $s \in (t_0, t)$ on the red line are calculated, and then the differential equation along the blue line can be calculated, yielding $p_{ij}(t_0, t, d, u)$.*

3. Continue calculating $(p_{ij}(t - nh, t, 0, u))_{i \in \mathcal{J}}$ for $n = 3, 4, \ldots$ as long as $t - nh \geq t_0$. In this way, we obtain the values $p_{ij}(s, t, 0, u)$ needed, that is, $i \in \mathcal{J}$ and $s \in [t_0, t)$, i.e. the red line on the first axis in Figure 1.

4. Finish with boundary condition $p_{ij}(t, t, d + t - t_0, u)$ and calculate $p_{ij}(t_0, t, d, u)$, where all the previously calculated values are used. Note, that if $d = 0$, this value was already calculated above with $n$ such that $t - nh = t_0$.

Now, we have calculated $p_{ij}(t_0, t, d, u)$ for $i \in \mathcal{J}$ and fixed $j$, $t$ and $u$. This algorithm must be repeated for each $j \in \mathcal{J}$ and $u \geq 0$, in order to get the distribution of $Z(t), U(t)$ conditional on $Z(t_0) = i, U(t_0) = d$. Doing this for all $t \geq t_0$ as well, the cash flow in Proposition 2.3 can be calculated.

If we count the number of calculation steps required to calculate the transition probabilities needed for the cash flow, we find the time complexity to be of order $\mathcal{O}(N^4)$. Intuitively, $N$ to the power of 4 is: one power for calculating one line, one power for all the lines, one power for all terminal durations, and one power for all terminal times. We see later that it can be of advantage to use Kolmogorov's forward integro-differential equation, which is presented below in Theorem 3.1 and is of order $\mathcal{O}(N^2)$. It should be noted, that it might be possible to optimise the backward algorithm such that it is $\mathcal{O}(N^3)$, by reusing results between the solutions for different terminal durations $u \geq 0$.

## 2.6 The Markov case

We consider the special case, where the transition rates do not depend on the duration, i.e. where $\mathbf{Z}$ is a Markov process. If also the payments do not depend on the duration, the duration dependent probabilities are not needed, and the formulae simplify. This corresponds to the classic setup, see e.g. [21] or [16]. For the transition probabilities, we write

$$p_{ij}(t,s) = P\left(Z(s) = j \middle| Z(t) = i\right).$$

Since nothing depends on the duration, we remove the dependency in the notation. Restating Proposition 2.3 in the Markov case, we get the cash flow and prospective reserve,

$$\begin{aligned} \mathrm{d}A_i(t,s) &= \sum_{j\in\mathcal{J}} p_{ij}(t,s) \left( \mathrm{d}B_j(s) + \sum_{\substack{k\in\mathcal{J}\\k\neq j}} \mu_{jk}(s) b_{jk}(s)\,\mathrm{d}s \right), \\ V_i(t) &= \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A_i(t,s). \end{aligned} \tag{2.7}$$

We now consider Proposition 2.4 in the Markov case, and obtain *Kolmogorov's backward differential equation for the Markov case*,

$$\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(s,t) = \mu_{i\cdot}(s) p_{ij}(s,t) - \sum_{\substack{k\in\mathcal{J}\\k\neq i}} \mu_{ik}(s) p_{kj}(s,t),$$

with boundary conditions $p_{ij}(t,t) = 1_{\{i=j\}}$.

When calculating the cash flow $(A_i(t_0,s))_{s\geq t_0}$ for fixed $t_0$, the transition probabilities $p_{ij}(t_0,s)$ need to be calculated for fixed $t_0$ and $s \geq t_0$. This can be done using Kolmogorov's backward differential equation. However, then one needs to solve the differential equation for each value of $s$, and one ends up with all the values $(p_{ij}(t,s))_{t\in(t_0,s)}$ for all $s \geq t_0$. It is then an advantage to use Kolmogorov's forward differential equation instead, which is a differential equation in $s$ instead of $t$. This yields all the needed transition probabilities immediately.

**Proposition 2.5.** (Kolmogorov's forward differential equation for the Markov case) *Let* $0 \leq t_0 \leq t$. *If the transition rates do not depend on the duration, the transition* *probabilities* $p_{ij}(t_0, t)$ *satisfy the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(t_0, s) = -p_{ij}(t_0, s)\mu_{j\cdot}(s) + \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq j}} p_{i\ell}(t_0, s)\mu_{\ell j}(s),$$
$$p_{ij}(t_0, t_0) = 1_{\{i=j\}}.$$

By using Kolmogorov's forward differential equation, the problem is, formally speaking, reduced by 1 dimension. In the next section, we present a generalised version of Kolmogorov's forward differential equation for the semi-Markov case, which does not seem to be well-known in the insurance literature.

# 3 Kolmogorov's forward integro-differential equation

In this Section we present a version of Kolmogorov's forward differential equation for the semi-Markov case, which seems less known than the differential equations for the Markov case presented above and which can also be found in textbooks on life insurance mathematics, see for example [16]. In the semi-Markov case, Kolmogorov's forward differential equation is an ordinary integro-differential equation, and it is similar to the partial differential equation obtained by [11], Corollary 2.37. The result can also be obtained from a similar integral equation presented in [14], by differentiation and some further calculations. As discussed in the previous section in the Markov case, using Kolmogorov's forward differential equation is preferable when calculating cash flows, since this leads directly to the necessary transition probability measures.

**Theorem 3.1.** (Kolmogorov's forward integro-differential equation) *Let* $0 \leq t_0 \leq t$ *and* $u \geq 0$ *and* $i \in \mathcal{J}$*. The transition probabilities*

$$p_{ij}\left(t_0, t, u, d + t - t_0\right), \qquad \text{for } j \in \mathcal{J} \text{ and } d \in \mathbb{R} \text{ s.t. } d + t - t_0 \geq 0,$$

*satisfy, with* $D(s) = d + s - t_0$*, the system of integro-differential equations given by*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}\left(t_0, s, u, D(s)\right) = &-\int_0^{D(s)} p_{ij}\left(t_0, s, u, \mathrm{d}z\right) \mu_{j\cdot}(s, z) \\ &+ \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq j}} \int_0^{u+s-t_0} p_{i\ell}\left(t_0, s, u, \mathrm{d}z\right) \mu_{\ell j}(s, z),\end{aligned} \tag{3.1}$$

*with boundary conditions* $p_{ij}\left(t_0, t_0, u, d\right) = 1_{\{i=j\}} 1_{\{u \leq d\}}$ *and, for* $s > t_0$*,* $p_{ij}\left(t_0, s, u, 0\right) = 0$.

For completeness we give a simple proof based on the Chapman-Kolmogorov equations and Kolmogorov's backward differential equations. The proof is presented in Appendix A.

The terms in the differential equation have a straightforward interpretation. The term on the first line of (3.1) corresponds to the transitions out of state $j$ at time $s$: It is the probability being in state $j$ at time $s$, and then leaving state $j$ at time $s$. The transition rate out of state $j$ is duration dependent, and thus the integral over the duration appears. We only integrate up to the current duration $D(s) = d + s - t_0$, as it is only probability mass with duration less than $d + s - t_0$ that is to be deducted. The term on the second line of (3.1) corresponds to new entries into state $j$ at time $s$. First, it is the probability of being in state $\ell$ at time $s$, where $\ell \neq j$, and then at time $s$ transitioning from state $\ell$ to state $j$. Again, this transition rate is duration dependent, and the integral appears. Here, we integrate up to the maximum duration $u + s - t_0$.

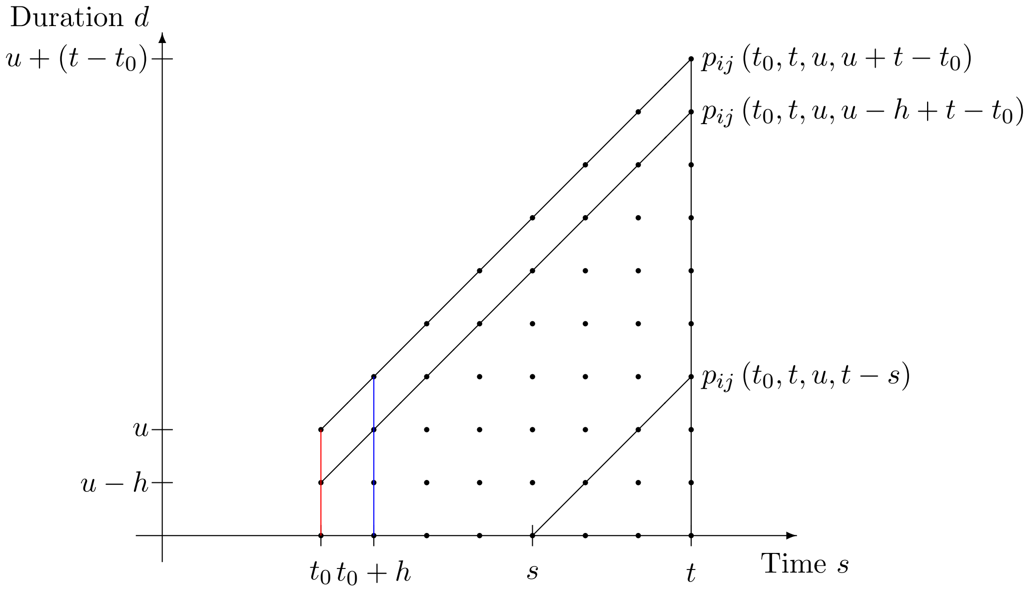


**Figure 2:** *Sketch of algorithm to calculate the transition probabilities* $(p_{ij}(t_0, s, u, d))_{j \in \mathcal{J},\, s \in [t_0, t],\, d \geq 0}$ *using Kolmogorov's forward integro-differential equation, using a numerical algorithm with step size h, matching the drawn grid. The distribution along the red line is given by the boundary conditions, and then the integro-differential equation is used to shift the distribution to the right, thus the distribution along the blue line is reached in 1 step, and continuing yields the distribution along the vertical black line.*

We illustrate how to calculate $p_{ij}(t_0, s, u, d)$ for all $j \in \mathcal{J}$, $s \in [t_0, t]$ and $d \in [0, u + s - t_0]$ where $t_0 \leq t$ and $u \geq 0$ and $i \in \mathcal{J}$ are fixed. Assume a grid of size $h$ according to Figure 2, and that $t_0, t$ and $u$ lie on the grid. We discretize the integral over the duration $d$

using this grid as well.

1. At time $t_0$, the boundary conditions yield the values $p_{ij}(t_0, t_0, u, d)$ for all $j$ and $d \in [0, u]$, corresponding to the distribution along the red line on Figure 2.
2. The differential equation (3.1) can now be used to calculate the probabilities $p_{ij}(t_0, t_0 + h, u, d + h)$, for all $j \in \mathcal{J}$ and $d \in (-h, u]$. Together with the new boundary condition at time $t_0 + h$ for duration 0, $p_{ij}(t_0, t_0 + h, u, 0)$, this yields the distribution along the blue line on Figure 2.

   In this step, note that the integral on the second line of (3.1) is independent of $d$, so the integral only needs to be solved once in this step. Also, the value of the integral on the first line can be reused for different durations: If the value is known for $d + s - t_0$, then for $d + h + s - t_0$, only the increment of the integral needs to be calculated.
3. Repeat this, calculating $p_{ij}(t_0, t_0 + nh, u, d + nh)$ for all $j \in \mathcal{J}$ and $d \in (-nh, u]$, for $n = 2, 3, \ldots$, until $t_0 + nh = t$ and the distribution along the vertical black line in Figure 2 is reached.

Note that the number of differential equations in the differential equation system is increased by one at each step, corresponding to the new diagonal line beginning at duration 0. The algorithm immediately yields all needed probabilities to calculate the cash flow in Proposition 2.3.

Compared to using the algorithm based on Kolmogorov's backward differential equation, as sketched in Figure 1, this is significantly simpler. The time complexity of solving Kolmogorov's forward integro-differential equations as described above, with the optimizations in step 2, is $\mathcal{O}(N^2)$. Intuitively, $N$ to the power of two is: one power for the time, and one power for the calculations needed for each time point (the two integrals and the advancement of the differential equation for each duration). This is simpler than the time complexity of $\mathcal{O}(N^4)$ when using Kolmogorov's backward differential equations for calculating cash flows.

## 4 Modelling of policyholder behaviour

We consider a life insurance contract in the above setup without modelling of policyholder behaviour, and show how to extend it to include policyholder behaviour modelling. Our modelling of the options as random transitions in a semi-Markov model implies that they are not necessarily exercised rationally, but instead randomly. Rational surrender is studied in [23], and attempts to couple rational and random behaviour are carried out in [9] and [4]. In this paper, we carry on with the modelling of policyholder behaviour as occuring randomly, and this corresponds to a large extent to the behaviour observed in practice.

The prospective reserve and the cash flow are calculated on different bases. First, we consider calculations on the so-called *technical basis*, with separate (and typically conservative) interest and transition rates, in a state space $\mathcal{J}$, without any policyholder modelling. The technical basis is used to determine the premiums as well as the surrender benefits and the benefits after a free policy conversion. In particular, the policyholder modelling may be omitted on the technical basis since the surrender and free policy benefits are chosen such that the value on the technical basis is unaffected. Second, we consider market consistent calculations on a so-called *market basis*. This is done with a market consistent interest rate structure and a best estimate of future transition rates. These calculations are used for the balance sheet values. The calculations can be conducted in the same state space $\mathcal{J}$, which is then without policyholder modelling. In this paper, we extend the state space in the market basis in order to include policyholder modelling, in the form of a surrender and a free policy option. For a comprehensive treatment of the policyholder options, see [20].

## 4.1 The life insurance contract and the technical basis

We consider a life insurance contract where the state of the insured is modelled by the semi-Markov process $\mathbf{Z}$ on the state space $\mathcal{J}$, where $\mathcal{J}$ does not include any policyholder modelling. The payment process $B(t)$ is decomposed into the benefit payments (the positive payments), described by $B^+(t)$, and the premium payments (the negative payments), described by $B^-(t)$, such that the total payments are $B(t) = B^+(t) - B^-(t)$. That is, the benefit payments process $B^+(t)$ is defined as the payment process (2.4) associated with the positive parts of the payment functions $b_i(t)^+, \Delta B_i(t)^+$ and $b_{ij}(t)^+$. Similarly $B^-(t)$ is the payment process of the negative parts.[4] The premiums are settled according to the equivalence principle on the technical basis consisting of a technical interest rate $r^* = (r^*(t))_t$ (which is usually constant) and a technical set of transition rates $\mu^* = (\mu^*_{ij}(t,u))_{i,j,t,u}$. For calculations on the technical basis, policyholder behaviour is not modelled.

The *technical reserve* of the life insurance contract is the prospective reserve calculated on the technical basis. We denote the technical reserve by $V^*_{i,u}(t)$, and it is defined similar to the prospective reserve in Definition 2.1, using the technical basis $(r^*, \mu^*)$. We decompose the technical reserve into the value associated with the benefits and the

[4] The notation with $+$ and $-$ is a bit ambiguous: For a function $f(t)$, we have $f(t)^+ = \max\{0, f(t)\}$, and if we write $f^+(t)$, it is merely a label.

value associated with the premiums, defining, for $i \in \mathcal{J}$, $u \geq 0$,

$$\begin{aligned} V_{i,u}^{*,+}(t) &= \mathrm{E}^* \left[ \int_t^\infty e^{-\int_t^s r^*(\tau)\,\mathrm{d}\tau} \,\mathrm{d}B^+(s) \middle| Z(t) = i, U(t) = u \right], \\ V_{i,u}^{*,-}(t) &= \mathrm{E}^* \left[ \int_t^\infty e^{-\int_t^s r^*(\tau)\,\mathrm{d}\tau} \,\mathrm{d}B^-(s) \middle| Z(t) = i, U(t) = u \right]. \end{aligned} \tag{4.1}$$

Thus, the technical reserve is given by $V_{i,u}^*(t) = V_{i,u}^{*,+}(t) - V_{i,u}^{*,-}(t)$. Here, the notation $E^*$ denotes expectation using the measure where $\mathbf{Z}$ has transition rates $\mu^*$.

## 4.2 The exercise options

We consider the problem of market based valuation of the life insurance contract including two exercise options of the policyholder. The policyholder options are:

- The *surrender* option, where the policyholder stops payments and receives the technical reserve.
- The *free policy* (paid-up policy) option, where the policyholder stops premium payments, and the benefits are reduced proportionally according to the equivalence principle on the technical basis.

We assume that the policyholder may exercise the surrender option at any time while in state 0, and similarly, we assume that the free policy option can only be exercised in state 0. For notational and interpretational simplicity, we assume that on the technical basis, there is no duration dependence in state 0. That is, we assume that for $t, u \geq 0$,

$$V_{0,u}^*(t) = V_{0,0}^*(t). \tag{4.2}$$

If one has duration dependence in state 0 on the technical basis, certain interpretational complications arise: One has to consider whether or not the duration should be reset to 0 after a free policy conversion, and thereby also whether or not the duration on the technical basis and on the market basis are identical.

### 4.2.1 The free policy option

If the policyholder exercises the free policy option at time $s$, the benefits are recalculated on the technical basis, according to the fact that that future premiums after time $s$ are cancelled. We define $\rho(s)$ as the benefit scaling factor if the policyholder exercises the free policy option at time $s$.

### 4.2.2 The free policy benefit factor

On the technical basis, the payment process, conditioning on the exercise of the free policy option at time $s$, is, for $t \geq s$,

$$t \mapsto \rho(s)\,\mathrm{d}B^{+}(t).$$

That is, the payments are the positive payments $\mathrm{d}B^{+}(t)$ scaled by the factor $\rho(s)$ at the time of free policy conversion. Then, the technical reserve at time $t \geq s$ in state $i \in \mathcal{J}$ is, using (4.1),

$$\rho(s) V_{i,u}^{*,+}(t). \tag{4.3}$$

We stress that this is on the technical basis, where the state space $\mathcal{J}$ is without policyholder modelling.

The equivalence principle is used here to determine $\rho(s)$, which here states that there is no jump in the technical reserve due to the a free policy conversion. Thus, we have the requirement $\rho(s)V_{0,0}^{*,+}(s) = V_{0,0}^{*}(s)$, which yields

$$\rho(s) = \frac{V_{0,0}^{*,+}(s) - V_{0,0}^{*,-}(s)}{V_{0,0}^{*,+}(s)}, \tag{4.4}$$

where we recall the assumption (4.2), which states that there is no duration dependence in state 0 for the technical reserve. In particular $\rho(s)$ is a deterministic function.

It should be noted, that the model of this paper can be used with other choices of $\rho(s)$, as long as it is a deterministic function. Also, we assumed that $V_{0,u}^{*}(t)$ is duration independent, and therefore $\rho(s)$ is duration independent. Proposition 4.1 and Theorem 4.2 below can readily be extended to allow for $\rho(s)$ being dependent on the duration immediately before the free policy conversion happened.

In practice there may be free policy options where the benefits do not scale proportionally as assumed here. For example, some benefits may be removed completely upon free policy conversion, and the rest may scale proportionally. In that case, for calculating the market value, the policy can be split into parts where $\rho$ may be different in each part. Then, the results of this paper can be used separately for each part of the policy.

### 4.2.3 The surrender option

If the policyholder surrenders the contract at time $t$, all future payments are cancelled, and instead the policyholder receives the current technical reserve. If the policy is still with premium, the technical reserve is $V_{0,0}^{*}(t)$, since surrender can only happen from state 0, and since there is no duration dependence on the technical basis in state 0. If the policy became a free policy at time $s$, the technical reserve at time $t$ is given by (4.3) with $i = u = 0$.

The surrender payment can be more general, and in particular it may depend on the duration at the time of surrender. Proposition 4.1 and Theorem 4.2 can readily be extended to allow for a duration dependent surrender payment, e.g. $V^*_{0,U(t-)}(t)$, if we had not assumed duration independence in state 0 on the technical basis.

## 4.3 Extending the Markov model

When valuing the life insurance contract on the market basis, we consider an extended state space, taking into account the policyholder exercise options. Thus, the difference between the technical basis and the market basis is not only the interest and transition rates, but also the state space and hence the payment process.

Consider the extended state space which includes policyholder behaviour (phb),

$$\mathcal{J}^{\mathrm{phb}} = \mathcal{J} \cup \mathcal{J}^{\mathrm{s}} \cup \mathcal{J}^{\mathrm{f}} \cup \mathcal{J}^{\mathrm{fs}},$$

see Figure 3. The original state space is $\mathcal{J} = \{0, 1, \ldots, J-1\}$, which is the one used for the technical basis. The new parts of the state space are;

- $\mathcal{J}^{\mathrm{s}} = \{J\}$, which contains the state for surrender when the free policy option is not exercised;
- $\mathcal{J}^{\mathrm{f}} = \mathcal{J} + (J+1) = \{J+1, J+2, \ldots, 2J\}$ which is a copy of $\mathcal{J}$, and is the states of the policy when the free policy option is exercised;
- $\mathcal{J}^{\mathrm{fs}} = \{2J+1\}$ which is the state for surrender as a free policy.

We assume that $\mathbf{Z}$ is a semi-Markov model on the extended state space $\mathcal{J}^{\mathrm{phb}}$, with the same assumptions as before, i.e. that the transition rates exist, defined by (2.2), and that $Z(0) = 0$. State 0 is the only state from which the free policy option can be exercised, and state 0 and $J+1$ are the only states from which the surrender option can be exercised. We say that the transition $0 \mapsto J+1$ is a free policy conversion, and that a transition $0 \mapsto J$ (or $J+1 \mapsto 2J+1$) is a surrender (as a free policy). The transition rates within $\mathcal{J}^{\mathrm{f}}$ are assumed to be identical to those of $\mathcal{J}$. The states in $\mathcal{J}^{\mathrm{s}}$ and $\mathcal{J}^{\mathrm{fs}}$ are absorbing.

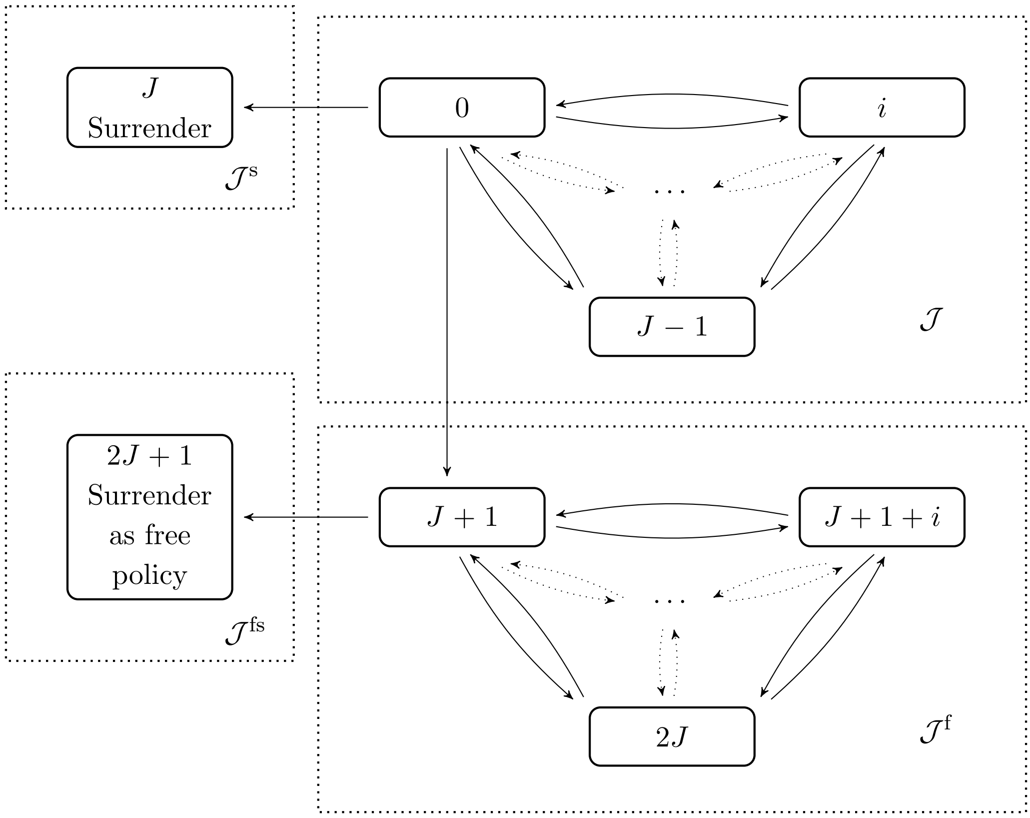


**Figure 3:** *A general finite state space $\mathcal{J}$ extended with free policy states $\mathcal{J}^{\mathrm{f}}$ and surrender states $\mathcal{J}^{\mathrm{s}}$ and $\mathcal{J}^{\mathrm{fs}}$. The state space $\mathcal{J}$ contains the state of the insured, e.g. active, disabled, dead, etc. The free policy states $\mathcal{J}^{\mathrm{f}}$ is a copy of $\mathcal{J}$, and a transition from $\mathcal{J}$ to $\mathcal{J}^{\mathrm{f}}$ corresponds to a free policy conversion. A transition to $J \in \mathcal{J}^{\mathrm{s}}$ or $2J+1 \in \mathcal{J}^{\mathrm{fs}}$ corresponds to a surrender of the policy.*

## 4.4 The payment process

For the total payment process in the extended model, we introduce a second duration process, $\mathbf{W}$, defined by

$$W(t) = \inf\{s \geq 0 | Z(t-s) \in \mathcal{J}\},$$

that is, $W(t)$ is the duration since $Z(t)$ left $\mathcal{J}$, or if $Z(t) \in \mathcal{J}$, then $W(t) = 0$. The duration $\mathbf{W}$ is required for the payments in the free policy states, because, if we are at time $t$ and $Z(t) \in \mathcal{J}^{\mathrm{f}}$, the free policy option was exercised at time $t - W(t)$, and we recall that $\rho(t - W(t))$ is used to determine the size of the benefits.

The payment process for the extended model can now be stated. On the original part of the state space, $\mathcal{J}$, the payment process is the same as on the technical basis, $B(t)$, and is defined by (2.4). On the market basis, the payment process on the extended state space is denoted $B^{\text{phb}}$, and is given by,

$$\begin{aligned}\mathrm{d}B^{\text{phb}}(t) = &\ \mathrm{d}B(t) + V^{*}_{0,0}(t)\,\mathrm{d}N_{0J}(t) \\ &+ \rho(t - W(t))\left(\mathrm{d}B^{\text{f},+}(t) + V^{*,+}_{0,0}(t)\,\mathrm{d}N_{J+1,2J+1}(t)\right).\end{aligned}$$

We briefly explain the payment process. For $Z(t) \in \mathcal{J}$, the payments are determined by the original payment process $B(t)$, that is, the original payment functions $b_j$, $\Delta B_j$ and $b_{ij}$, for $i, j \in \mathcal{J}$. By Section 4.2.3, the payment upon surrender as a premium paying policy is $V^{*}_{0,0}(t)$, and this is paid upon a transition from state 0 to state $J$. As a free policy, the payment upon surrender is $\rho(t - W(t))V^{*,+}_{0,0}(t)$, which is paid upon a transition from state $J+1$ to state $2J+1$. The payment process $B^{\text{f},+}(t)$ is non-zero for $Z(t) \in \mathcal{J}^{\text{f}}$ and is the payments as a free policy, excluding the scaling factor $\rho(t-W(t))$. Intuitively $B^{\text{f},+}(t)$ is a copy on $\mathcal{J}^{\text{f}}$ of the positive payments on $\mathcal{J}$, i.e. of $B^{+}(t)$. To be precise, define the payment functions $b_i$, $\Delta B_i$ and $b_{ij}$ on $\mathcal{J}^{\text{f}}$, by simply copying their value on $\mathcal{J}$, i.e. for $i, j \in \mathcal{J}^{\text{f}}$ let,

$$b_i = b_{i-(J+1)}, \quad \Delta B_i = \Delta B_{i-(J+1)}, \quad b_{ij} = b_{i-(J+1),j-(J+1)}.$$

Then $B^{\text{f},+}(t)$ satisfies

$$\begin{aligned}\mathrm{d}B^{\text{f},+}(t) &= \sum_{i\in\mathcal{J}^{\text{f}}} 1_{\{Z(t)=i\}}\,\mathrm{d}B^{+}_{i}(t, U(t)) + \sum_{\substack{i,j\in\mathcal{J}^{\text{f}}\\ i\neq j}} b_{ij}(t, U(t-))^{+}\,\mathrm{d}N_{ij}(t), \\ \mathrm{d}B^{+}_{i}(t, U(t)) &= b_i(t, U(t))^{+}\,\mathrm{d}t + \Delta B_i(t, U(t))^{+}.\end{aligned}$$

## 4.5 Cash flows with policyholder behaviour

We now study the cash flow associated with the payment process $\mathbf{B}^{\text{phb}}$ in the extended model. The cash flow at time $t$ for the future payments is denoted $(A^{\text{phb}}_{i,u}(t,s))_{s\geq t}$. By the use of transition probabilities dependent on the two durations, $\mathbf{U}$ and $\mathbf{W}$, we can find the cash flow. We note that $\mathbf{Z}$ is a semi-Markov process such that $(\mathbf{Z}, \mathbf{U})$ is a Markov process, but the payments may depend on the additional duration $\mathbf{W}$.

Due to the simple structure for the extended model, the cash flow for state $i \in \mathcal{J}$ may be determined directly via a prospective argument. In this way, it can be shown that the cash flow satisfies, for $i \in \mathcal{J}$, $u \geq 0$ and $s \geq t$,

$$\mathrm{d}A^{\text{phb}}_{i,u}(t,s) = \mathrm{d}A_{i,u}(t,s) + \int_{0}^{u+s-t} p_{i0}\,(t, s, u,\,\mathrm{d}z)\,\mu_{0J}(s,z)V^{*}_{0,0}(s)\,\mathrm{d}s$$

$$
\begin{aligned}
&+\sum_{j\in\mathcal{J}^{\mathrm{f}}}\int_0^\infty\int_0^\infty P(Z(s)=j,U(s)\le\,\mathrm{d}z,W(s)\le\,\mathrm{d}w|Z(t)=i,U(t)=u)\\
&\quad\times\rho(s-w)\left(\mathrm{d}B_j^+(s,z)+\sum_{\substack{k\in\mathcal{J}^{\mathrm{f}}\\k\neq j}}\mu_{jk}(s,z)b_{jk}(s,z)^+\,\mathrm{d}s\right)\\
&+\int_0^\infty\int_0^\infty P(Z(s)=J+1,U(s)\le\,\mathrm{d}z,W(s)\le\,\mathrm{d}w|Z(t)=i,U(t)=u)\\
&\quad\times\rho(s-w)\mu_{J+1,2J+1}(s,z)V_{0,0}^{*,+}(s)\,\mathrm{d}s.
\end{aligned}
$$

Here, the first part of the first line is the insurance payments $\mathrm{d}A_{i,u}(t,s)$, which is the cash flow of the original payment process $B(s)$, i.e. the payments while in $\mathcal{J}$. Also, the second part of the first line are payments upon surrender from state 0. The second and third lines represent the payments of the payment process $B^{\mathrm{f},+}$, i.e. as a free policy. The last lines are surrenders as a free policy. In the last term, we have used that the first order reserve at time $s$ in the free policy state $J+1$ is equal to $\rho(s-w)V_{0,0}^{*,+}(s)$ if the free policy option was exercised at time $s-w$.

The cash flow $\mathrm{d}A_{i,u}(t,s)$ is given by

$$
\mathrm{d}A_{i,u}(t,s)=\sum_{j\in\mathcal{J}}\int_0^{u+s-t}p_{ij}\left(t,s,u,\,\mathrm{d}z\right)\left(\mathrm{d}B_j(s,z)+\sum_{\substack{k\in\mathcal{J}\\k\neq j}}\mu_{jk}(s,z)b_{jk}(s,z)\,\mathrm{d}s\right).
$$

The transition probabilities can be calculated using Kolmogorov's forward integro-differential equation, (3.1), which in this case is on the state space $\mathcal{J}^{\mathrm{phb}}$. In particular, the cash flow $A_{i,u}(t,s)$ differs from the one obtained without policyholder modelling where the transition probabilities in the smaller semi-Markov model with state space $\mathcal{J}$ are used.

Because of the structure of the semi-Markov model, knowledge of $\mathbf{W}$ is the same as knowledge of the time of transition from state 0 to state $J+1$, and there can only be one such transition. Recall that by the design, a transition from state 0 to state $J+1$ is necessary to go from state $i\in\mathcal{J}$ to state $j\in\mathcal{J}^{\mathrm{f}}\cup\mathcal{J}^{\mathrm{fs}}$. With this, it can be shown that, for $i\in\mathcal{J}$ and $j\in\mathcal{J}^{\mathrm{f}}\cup\mathcal{J}^{\mathrm{fs}}$,

$$
\begin{aligned}
&\int_0^\infty P(Z(s)=j,U(s)\le z,W(s)\le\,\mathrm{d}w|Z(t)=i,U(t)=u)\rho(s-w)\\
&=\int_t^s\int_0^{u+\tau-t}p_{i0}\left(t,\tau,u,\,\mathrm{d}v\right)\mu_{0,J+1}(\tau,v)\rho(\tau)p_{J+1,j}\left(\tau,s,0,z\right)\,\mathrm{d}\tau\\
&=:p_{ij}^{\rho}\left(t,s,u,z\right).
\end{aligned}
\tag{4.5}
$$

Here we defined $p^\rho$, and it is the probability, given we start in state $i \in \mathcal{J}$ at time $t$ with duration $u$, of being in state $j \in \mathcal{J}^{\mathrm{f}} \cup \mathcal{J}^{\mathrm{fs}}$ with duration less than $z$ at time $s$, multiplied by $\rho$ at the time of transition from state 0 to $J+1$. Using the quantities $p^\rho$, we can rewrite the cash flow in the following way.

**Proposition 4.1.** *The cash flow $(A^{\mathrm{phb}}(t,s))_{s\geq t}$ at time $t$, with policyholder behaviour, is given by*

$$\begin{aligned}
\mathrm{d}A^{\mathrm{phb}}_{i,u}(t,s) = \; & \mathrm{d}A_{i,u}(t,s) + \int_0^{u+s-t} p_{i0}\left(t,s,u,\,\mathrm{d}z\right)\mu_{0J}(s,z)V^*_{0,0}(s)\,\mathrm{d}s \\
& + \sum_{j\in\mathcal{J}^{\mathrm{f}}} \int_0^{u+s-t} p^\rho_{ij}\left(t,s,u,\,\mathrm{d}z\right)\left(\mathrm{d}B^+_j(s,z) + \sum_{\substack{k\in\mathcal{J}^{\mathrm{f}}\\ k\neq j}} \mu_{jk}(s,z)b_{jk}(s,z)^+\,\mathrm{d}s\right) \\
& + \int_0^{u+s-t} p^\rho_{i,J+1}\left(t,s,u,\,\mathrm{d}z\right)\mu_{J+1,2J+1}(s,z)V^{*,+}_{0,0}(s)\,\mathrm{d}s,
\end{aligned}$$

*for $i \in \mathcal{J}$, where $p^\rho_{ij}(t,s,u,z)$ is defined by* (4.5)*.*

The first part of the first line is the cash flow when none of the policyholder options has been exercised. The second part of the first line is the payment upon surrender when a free policy transition has not occured. The second line contains the payments after a free policy transition, and the last line is the payment associated with an exercise of the surrender option after a free policy conversion.

In order to calculate the cash flow, one needs to determine $p^\rho_{ij}(t,s,u,z)$. This is time-consuming compared to the model without policyholder behaviour, because the extra probabilities $p_{J+1,j}(\tau,s,0,z)$ in (4.5) need to be calculated for all $\tau \in (t,s)$ and $s \geq t$. That is, in excess of just solving Kolmogorov's forward integro-differential equation, we now need to solve for each $s \geq t$ as well. Formally, we can say that the dimension of the problem is doubled if we determine $p^\rho$ directly via (4.5).

See, that if $\rho(t) = 1$ for all $t$, then, for $i \in \mathcal{J}$ and $j \in \mathcal{J}^{\mathrm{f}} \cup \mathcal{J}^{\mathrm{fs}}$, we have $p^\rho_{ij}(t,s,u,z) = p_{ij}(t,s,u,z)$. In that case, we can use Kolmogorov's forward integro-differential equation directly, which are significantly simpler than calculating (4.5). However, it turns out, that for $\rho(t) \neq 1$, we can find a similar forward integro-differential equation system for $p^\rho$, which is the main result of this paper.

**Theorem 4.2.** *Let $0 \leq t_0 \leq t$ and $u \geq 0$ and $i \in \mathcal{J}$. The quantities*

$$p^\rho_{ij}\left(t_0,t,u,d+t-t_0\right), \quad \text{for } j \in \mathcal{J}^{\mathrm{f}} \cup \mathcal{J}^{\mathrm{fs}} \text{ and } d \in \mathbb{R} \text{ s.t. } d+t-t_0 \geq 0,$$

*satisfy, with $D(s) = d + s - t_0$, the forward system of integro-differential equations,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} p^{\rho}_{ij}\left(t_0, s, u, D(s)\right) = 1_{\{j=J+1\}} &\int_0^{u+s-t_0} p_{i0}\left(t_0, s, u, \,\mathrm{d}z\right) \mu_{0,J+1}(s,z)\rho(s) \\ &- \int_0^{D(s)} p^{\rho}_{ij}\left(t_0, s, u, \,\mathrm{d}z\right) \mu_{j\cdot}(s,z) \\ &+ \sum_{\substack{\ell\in\mathcal{J}^{\mathrm{f}}\\ \ell\neq j}} \int_0^{u+s-t_0} p^{\rho}_{i\ell}\left(t_0, s, u, \,\mathrm{d}z\right) \mu_{\ell j}(s,z),\end{aligned} \tag{4.6}$$

*with boundary conditions $p^{\rho}_{ij}\left(t_0, t_0, u, d\right) = 0$ and, for $s \geq t_0$, $p^{\rho}_{ij}\left(t_0, s, u, 0\right) = 0$.*

The first line of the differential equation corresponds to new free policy conversions, i.e. transitions from state 0 to state $J+1$. We see that here the transition rate is multiplied with the free policy factor $\rho$. The two other lines correspond to the differential equation from Kolmogorov's forward integro-differential equations, Theorem 3.1. Line two is transitions away from state $j$ at time $s$, where the duration is less than the current duration $D(s) = d + s - t$. The last line, line three, is transitions from other states in $\mathcal{J}^{\mathrm{f}}$ to state $j$, for any duration.

*Proof of Theorem 4.2.* Differentiate $p^{\rho}$, (4.5), and apply Kolmogorov's forward integro-differential equations (Theorem 3.1), to obtain the integro-differential equation,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s} p^{\rho}_{ij}\left(t, s, u, D(s)\right) \\ &= \int_0^{u+s-t} p_{i0}\left(t, s, u, \,\mathrm{d}v\right) \mu_{0,J+1}(s,v)\rho(s) p_{J+1,j}\left(s, s, 0, D(s)\right) \\ &\quad + \int_t^s \int_0^{u+\tau-t} p_{i0}\left(t, \tau, u, \,\mathrm{d}v\right) \mu_{0,J+1}(\tau,v)\rho(\tau) \frac{\mathrm{d}}{\mathrm{d}s} p_{J+1,j}\left(\tau, s, 0, D(s)\right) \mathrm{d}\tau \\ &= 1_{\{j=J+1\}} \int_0^{u+s-t} p_{i0}\left(t, s, u, \,\mathrm{d}v\right) \mu_{0,J+1}(s,v)\rho(s) \\ &\quad + \int_t^s \int_0^{u+\tau-t} p_{i0}\left(t, \tau, u, \,\mathrm{d}v\right) \mu_{0,J+1}(\tau,v)\rho(\tau) \Bigg( \\ &\qquad - \int_0^{D(s)} p_{J+1,j}\left(\tau, s, 0, \,\mathrm{d}z\right) \mu_{j\cdot}(s,z) \,\mathrm{d}\tau \\ &\qquad + \sum_{\substack{\ell\in\mathcal{J}^{\mathrm{f}}\\ \ell\neq j}} \int_0^{s-\tau} p_{J+1,\ell}\left(\tau, s, 0, \,\mathrm{d}z\right) \mu_{\ell j}(s,z) \,\mathrm{d}\tau \Bigg)\end{aligned}$$

$$
\begin{aligned}
&= 1_{\{j=J+1\}} \int_0^{u+s-t} p_{i0}\,(t,s,u,\,\mathrm{d}v)\,\mu_{0,J+1}(s,v)\rho(s) \\
&\quad - \int_0^{D(s)} p^{\rho}_{ij}\,(t,s,u,\,\mathrm{d}z)\,\mu_{j\cdot}(s,z)\,\mathrm{d}\tau + \sum_{\substack{\ell\in\mathcal{J}^{\mathrm{f}}\\ \ell\neq j}} \int_0^{u+s-t} p^{\rho}_{i\ell}\,(t,s,u,\,\mathrm{d}z)\,\mu_{\ell j}(s,z)\,\mathrm{d}\tau.
\end{aligned}
$$

Here, the last equality is obtained by interchanging the order of integration. The boundary condition is obtained directly from (4.5). □

Considering the integro-differential equation (4.6) for $p^{\rho}$, we see that it is of similar structure as Kolmogorov's forward integro-differential equations, and that (4.6) depends on $p_{i0}\,(t,s,u,z)$. Thus, one can with advantage solve for $p$ and $p^{\rho}$ simultaneously, by solving the combined system of integro-differential equations, (3.1) and (4.6). Whereas, by letting the payments depend on $\rho$ at the time of the free policy transition essentially creates a double-duration setup, Theorem 4.2 eliminates one duration. Thus, computationally, the calculation of cash flows in the model with the two policyholder options is almost as simple as without: one simply needs to solve a system of integro-differential equations, and no extra duration is effectively introduced.

## 4.6 Policyholder behaviour in the Markov case

We consider the policyholder behaviour model for the special case where there is no duration dependence, i.e. neither the transition rates nor the payment functions depend on $\mathbf{U}$. When the transition rates do not depend on the duration, $\mathbf{Z}$ is a Markov process, and the setup is similar to Section 2.6. We still model the free policy option, so the payment process is dependent on the free policy duration $\mathbf{W}$.

First, $p^{\rho}$ simplifies a bit, since the integration of the duration disappears, and we have,

$$
p^{\rho}_{ij}(t,s) = \int_t^s p_{i0}(t,\tau)\mu_{0,J+1}(\tau)\rho(\tau)p_{J+1,j}(\tau,s)\,\mathrm{d}\tau. \tag{4.7}
$$

Then, the cash flow valuated at time $t$, given we are in state $i \in \mathcal{J}$, becomes

$$
\begin{aligned}
\mathrm{d}A^{\mathrm{phb}}_i(t,s) = {}& \mathrm{d}A_i(t,s) + p_{i0}(t,s)\mu_{0J}(s)V^{*}_0(s)\,\mathrm{d}s \\
& + \sum_{j\in\mathcal{J}^{\mathrm{f}}} p^{\rho}_{ij}(t,s)\left(\mathrm{d}B^{+}_j(s) + \sum_{\substack{k\in\mathcal{J}^{\mathrm{f}}\\ k\neq j}} \mu_{jk}(s)b_{jk}(s)^{+}\,\mathrm{d}s\right) \\
& + p^{\rho}_{i,J+1}(t,s)\mu_{J+1,2J+1}(s)V^{*,+}_0(s)\,\mathrm{d}s,
\end{aligned}
$$

where $\mathrm{d}A_i(t,s)$ is given in (2.7), using the state space $\mathcal{J}^{\text{phb}}$. The integro-differential equation for $p^\rho$ reduces to a simple differential equation. For $i \in \mathcal{J}$ and $j \in \mathcal{J}^{\text{f}} \cup \mathcal{J}^{\text{fs}}$, $p^\rho$ satisfies the forward differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}s} p^\rho_{ij}(t,s) = 1_{\{j=J+1\}} p_{i0}(t,s)\mu_{0,J+1}(s)\rho(s) - p^\rho_{ij}(t,s)\mu_{j\cdot}(s) + \sum_{\substack{\ell \in \mathcal{J}^{\text{f}} \\ \ell \neq j}} p^\rho_{i\ell}(t,s)\mu_{\ell j}(s), \quad (4.8)$$

with boundary condition $p^\rho_{ij}(t,t) = 0$.

From the differential equation for $p^\rho$, it is easy to see, that if $\rho(t) = 1$ for all $t$, then $p^\rho_{ij} = p_{ij}$ for $i \in \mathcal{J}$ and $j \in \mathcal{J}^{\text{f}} \cup \mathcal{J}^{\text{fs}}$. For $\rho(t) \neq 1$, we can interpret the differential equation intuitively. The second term of the right hand side corresponds to transitions out of state $j$, i.e. probability mass leaving state $j$. The third term corresponds to probability mass entering state $j$ from one of the other free policy states $\ell \in \mathcal{J}^{\text{f}}$. The first term then corresponds to new free policy conversions, i.e. transitions into $\mathcal{J}^{\text{f}}$, and this probability mass is multiplied by $\rho$. Thus, the quantities $p^\rho$ can be interpreted as manipulated transition probabilities in the way that probability mass that enters $\mathcal{J}^{\text{f}}$ is manipulated through a modified transition rate.

### 4.6.1 Connection with retrospective reserves

The definition of $p^\rho_{ij}(t,s)$ in (4.7) can be interpreted as a form of retrospective reserve, where the interest rate is set to 0. To see this, consider the state space from Figure 3 and associate a single payment $(-\rho(t))$ upon transition from state 0 to state $J+1$. Then the retrospective reserve $W^-_j(t)$ in Section 5 E of [21] equals $p^\rho_{0j}(0,t)$ from (4.7), if the interest rate is set to 0. In particular, (4.8) equals the differential equation system (5.14) in [21] with $W^-_j(t) = 0$ for $j \in \mathcal{J} \cup \mathcal{J}^{\text{s}}$.

To understand that a transitional payment of $(-\rho(t))$ is correct, note, that if an individual is in a free policy state now, e.g. state $J+1$, the value of the policy is the accumulated previous premiums, which is exactly $\rho(t - W(t))$. Then, a retrospective reserve for state $J+1$, with zero interest rate, is

$$p_{0,J+1}(0,t)\,\mathrm{E}\left[\rho(t-W(t))\middle|\, Z(t) = J+1\right] = \mathrm{E}\left[\rho(t-W(t))1_{\{Z(t)=J+1\}}\right],$$

which is exactly $p^\rho_{0,J+1}(0,t)$, where $i \in \mathcal{J}$.

For a similar connection in the semi-Markov setup, one can compare with Chapter 5 in [11]: Using (4.5), one can see that for $a \in \mathcal{J}$ and $y \in \mathcal{J}^{\text{f}} \cup \mathcal{J}^{\text{fs}}$, the quantity $\frac{p^\rho_{ay}(0,t,0,v)}{p_{ay}(0,t,0,v)}$ is analogous to (5.2.1) in [11].

## 5 Numerical example

We consider a numerical example and study how the modelling of policyholder behaviour has an effect on the liabilities. Specifically we consider the cash flow and the interest rate sensitivity in form of the dollar duration, and the prospective reserve. We show that introducing policyholder behaviour in the model significantly changes the structure of the cash flows, and to an extent also the prospective reserve.

For simplicity, we consider a survival model, i.e. a 2-state Markov model with states 0, *alive*, and 1, *dead*. We consider an insured male of age 40 with pension age 65, and two products,

1. a life annuity, starting at age 65,
2. a 10 year annuity upon death, if death occurs before age 65.

We assume that the insured has already saved an amount of 100,000, and that he pays a yearly premium of 10,000 while alive, until age 65. We specify the payment functions from (2.4),

$$
\begin{aligned}
b_0(t,u) &= 37{,}404 \cdot 1_{\{t \geq 25\}}, \\
b_1(t,u) &= 18{,}702 \cdot 1_{\{t-u<25\}} 1_{\{u<10\}}.
\end{aligned}
$$

The life annuity is of size 37,404, and the annuity upon death corresponds to 50% of the life annuity. On the technical basis introduced below, it gives a technical reserve of 100,000, which is the already saved up amount.

The annuity upon death is dependent on the duration $u$, since it is only for the 10 first years after death that there is a payment. This implies that we need the transition probabilities with durations from Theorem 3.1, instead of the simpler ones from Proposition 2.5. In other words, we are in practice in the semi-Markov setup, even though the transition rates are not duration dependent and $\mathbf{Z}$ is a Markov process.

We consider valuation on a technical basis, as well as three different market bases. The technical basis is the one used for pricing, and it is the value on the technical basis, $V^*(t)$ that is paid out upon surrender. Quantities related to the technical basis are generally marked with $*$. A market basis is used for calculating market consistent values, i.e. the prospective reserve for the balance sheet. It consists of a market interest rate and a best estimate of the mortality rate. We consider three different market bases, with three different Markov models: One without policyholder behaviour modelling, one with surrender modelling only, and one with surrender and free policy modelling. The three different market bases are used to illustrate the effect of including policyholder modelling.

The technical basis consists of the following,

- a survival Markov model with 2 states, *alive* and *dead*,
- an interest rate, $r^*(t) = 0.015$,
- a mortality rate given by the Danish G82M mortality table, $\mu^*(x) = 0.0005 + 0.000075858 \cdot 1.09144^x$.

This technical basis has been used historically, and a more conservative technical basis would be used for new contracts. As stated above, the benefits on our policy are consistent with a technical reserve of 100,000, calculated using (2.5).

Common for the three market bases is the interest and mortality rates. The interest rate is the one published every day by the Danish FSA for discounting life insurance liabilities, and the one from 8 May 2013 is used; it is available at [7]. The mortality rate is the benchmark mortality rate for 2011, which is published by the Danish FSA; it is available at [8]. See [15] for a further treatment of this mortality benchmark.

The difference between the three market bases is the Markov model, where the first model is the survival model without any policyholder modelling, shown in Figure 4a. The second model is the 3-state model, where the survival model is extended with a surrender state, according to Figure 4b. The third model is the 6-state model including free policy modelling and surrender modelling, see Figure 4c. The surrender transition rate $\mu^{\mathrm{s}}(x)$ and the free policy transition rate $\mu^{\mathrm{f}}(x)$ are for $x \leq 65$ given by,

$$\mu^{\mathrm{s}}(x) = 0.06 - 0.002 \cdot (x - 40)^+,$$
$$\mu^{\mathrm{f}}(x) = 0.05,$$

where $x$ is the age. We assume that surrender and free policy conversion cannot occur after age 65, and thus the rates are zero above age 65. The surrender and free policy parameters loosely resemble the ones used in practice by a large Danish life insurer in the competitive market. The transition rates are shown in Figure 5 together with the corresponding transition probabilities from the 6-state model in Figure 4c. The probability of surrender and free policy are significant and already at age 47 the probability of either surrender or free policy is higher than the probability of still being a premium paying policy.

To find the free policy factor $\rho(t)$ from (4.4), the technical reserve is calculated for all future time points. This yields a first value $\rho(0) = 0.34$, and then it increases almost linearly, slightly concave, to $\rho(25) = 1$, at pension age 65.

On the three market bases we consider the cash flows. In Figure 6 the premiums and benefits cash flow are shown for the three models, and in Figure 7, the surrender payments and the total cash flow are shown. For the premiums, we see that introducing surrender greatly reduces the amount of premiums, and a free policy conversion reduces

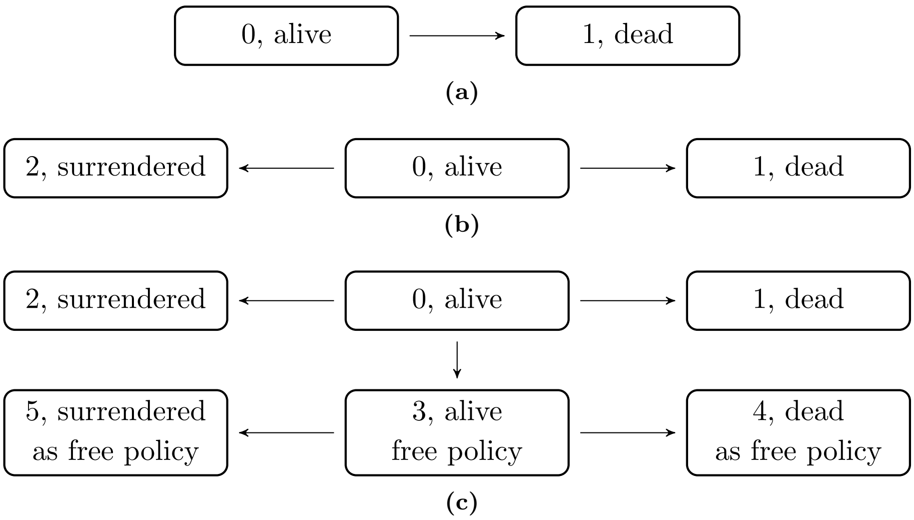


**Figure 4:** *The three state spaces for the market bases. In (a) there is no policyholder modelling. In (b) the surrender behaviour is modelled, and in (c) both the surrender and free policy behaviour is modelled.*

the premiums further. This is as expected, since if either a surrender or a free policy conversion occurs, future premiums are cancelled. For the benefits we see the same effect as for the premiums, where the introduction of both surrender and free policy modelling greatly reduces the benefits. Some of the reduction of the benefits due to the surrender modelling corresponds to value that is being paid out as surrender payments, shown in Figure 7, and some corresponds to the value of the future premiums that are cancelled upon surrender. The small benefits before age 65 are related to the annuity upon death.

The total cash flows in Figure 7 are the sum of the premiums, benefits and surrender payments and show the overall differences. Without policyholder modelling, there are first 25 years of negative payments, the premiums, and then after the pension age, the saved up value is paid out. If policyholder behaviour is included, the cash flows are in general smaller, and some of the benefits are also paid out earlier as surrender payments.

| | Basic | Surrender | Sur. and free pol. |
|---|---|---|---|
| Prospective reserve | 105,185 | 101,536 | 101,371 |
| DV01 Total | 83,407 | 41,838 | 34,547 |
| DV01 Pos. payments | 105,673 | 57,535 | 43,085 |
| DV01 Premiums | 22,266 | 15,697 | 8,538 |

**Table 1:** *Prospective reserves and dollar durations (DV01). The duration is greatly reduced when policyholder modelling is included, both for the total cash flows and also for the positive payments and premiums separately.*

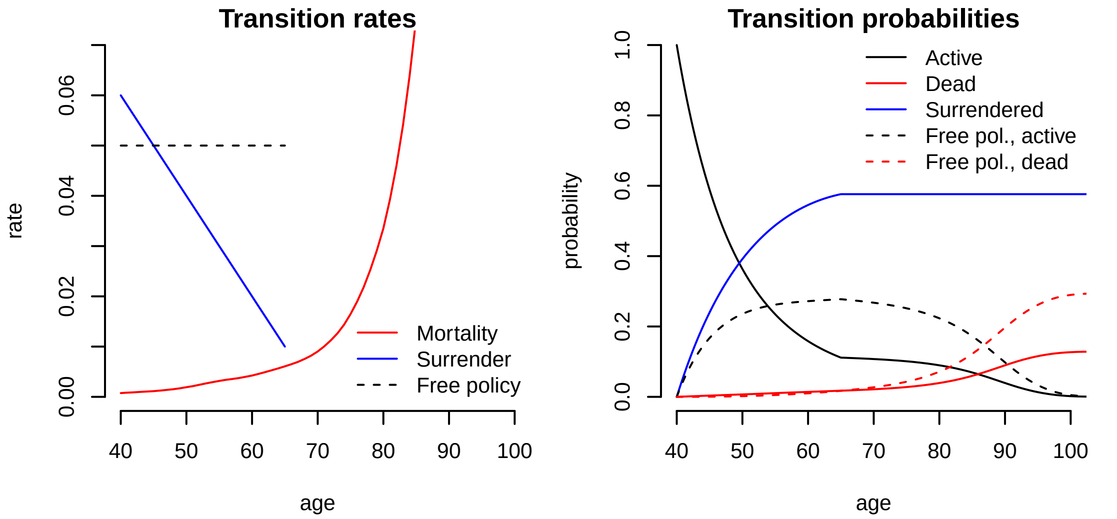


**Figure 5:** *Transition rates for the market basis (left) and transition probabilities in the 6-state surrender and free policy Markov model (right). The mortality rate is for a 40 year old male and shown including future improvements in mortality. The surrender rate is decreasing, and the free policy rate is constant. After age 65 it is not possible to surrender or convert to a free policy.*

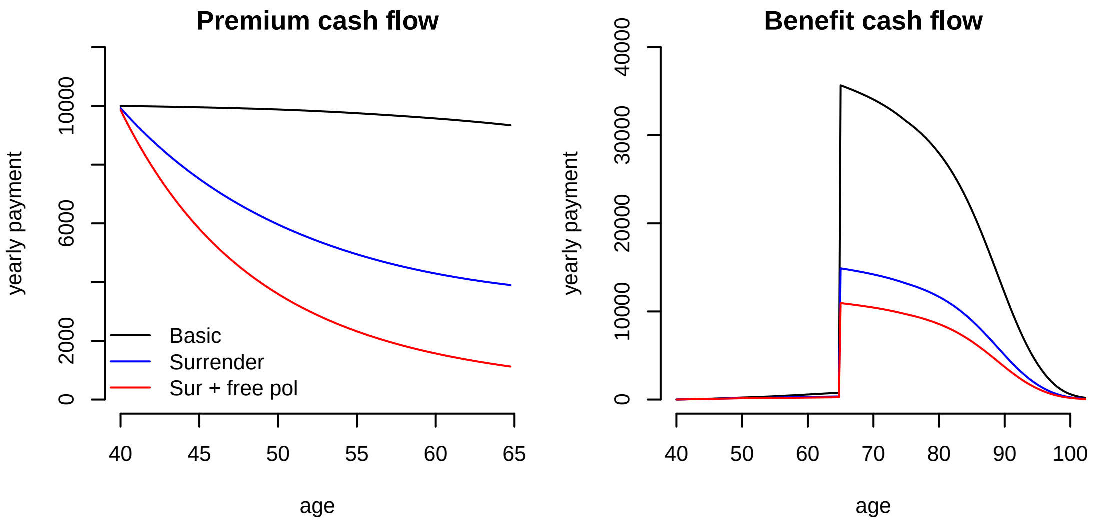


**Figure 6:** *Cash flow of premiums (left) and benefits (right). The premium cash flow in the basic model is slowly decreasing due to a small probability of death. With policyholder modelling, the premium cash flow is greatly reduced, since premiums are cancelled when either surrender or a free policy conversion occurs. The benefits are slightly increasing before age 65, due to a small probability of the annuity upon death being paid out. At age 65 the life annuity begins. Surrender and free policy modelling greatly reduces the benefits cash flow, since the benefits are reduced by a factor $\rho(t)$ upon transition to a free policy, and cancelled upon surrender.*

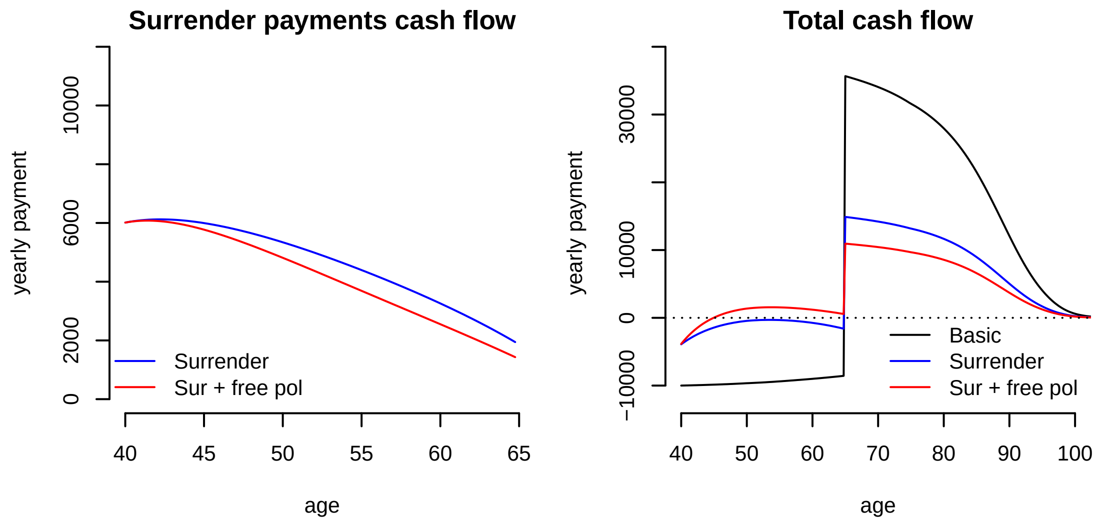


**Figure 7:** *Cash flow of surrender payments (left) and the total cash flow (right). There are no surrender payments in the basic model. With free policy modelling, the surrender payments are slightly smaller, since surrender as free policy yields a smaller payment. The total cash flows are the sum of the premium, benefit and surrender cash flows, and show that the payments, both positive and negative, are significantly reduced.*

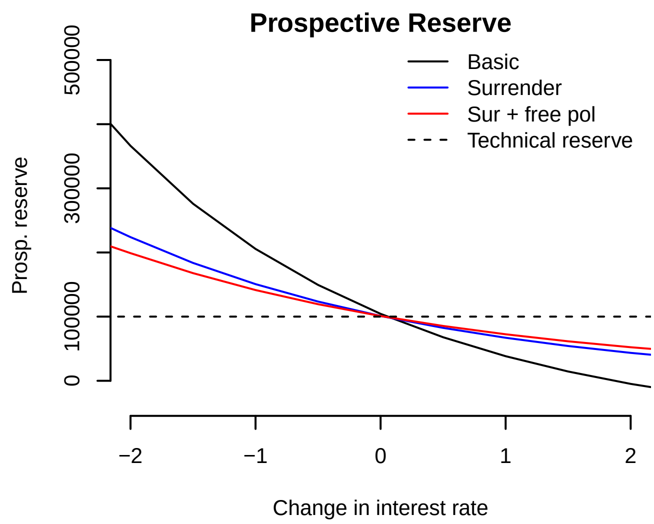


**Figure 8:** *Prospective reserves for the life insurance liabilities shown for different parallel shifts of the interest rate structure. The interest rate is capped at 0, so it is always non-negative. It is seen that modelling of policyholder behaviour reduces the interest rate sensitivity of the liabilities.*

The prospective reserves can be calculated with a market interest rate, and here we use the one provided by the Danish FSA of 8 May 2013. The prospective reserves are shown in Table 1 and it is seen that the value does not change a lot when introducing policyholder modelling. However, this change is very interest rate dependent, and in Figure 8, the prospective reserves are shown for different shifts in the market interest rate. We stress that the lines do not cross in zero: The technical reserve is independent of the market interest and equal to 100,000, and as seen in Table 1, the market values in the three models are slightly different from 100,000. When the market interest rate changes it is seen from the figure that the value in the basic model without policyholder behaviour changes the most. With surrender modelling, the market value changes less, and even less with free policy modelling. Thus, we see that policyholder modelling greatly influences the interest rate sensitivity.

In Table 1 the interest rate sensitivity for the total cash flows is shown, in form of the dollar duration, DV01, which measures the absolute change in the value for a 100 basis point change in the interest rate. Surrender modelling reduces the dollar duration with about 50% and on top of that free policy modelling reduces the duration by another 17%. Thus, if one applies duration matching for hedging the interest rate risk, it is essential to take into account both surrender and free policy modelling.

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# A Proof of Kolmogorov's forward integro-differential equation

Before we can prove Theorem 3.1, we recall the Chapman-Kolmogorov equation.

**Lemma A.1.** (The Chapman-Kolmogorov equation) *Let $t_0 \leq s \leq t$ and $u, z \geq 0$. The transition probabilities satisfy*

$$p_{ij}\left(t_0, t, u, z\right) = \sum_{\ell \in \mathcal{J}} \int_0^{u+s-t_0} p_{i\ell}\left(t_0, s, u, \,\mathrm{d}v\right) p_{\ell j}\left(s, t, v, z\right).$$

In the proof below, we use a slightly rewritten version, where, for $0 \leq h \leq t - t_0$, we write the Chapman-Kolmogorov equation as,

$$\begin{aligned} &p_{ij}\left(t_0, t, u, z\right) \\ &= \sum_{\ell \in \mathcal{J}} \int_0^{u+t-t_0} p_{i\ell}\left(t_0, t-h, u, \,\mathrm{d}v - h\right) p_{\ell j}\left(t-h, t, v-h, z\right), \end{aligned} \tag{A.1}$$

using the notation

$$p_{i\ell}\left(t_0, t-h, u, \mathrm{d}v-h\right) = P(Z(t-h)=\ell, U(t-h)+h \leq \mathrm{d}v | Z(t_0)=i, U(t_0)=u).$$

We also recall a result on weak convergence of measures, which is stated as Theorem 2.1 in [2].

**Lemma A.2.** *Let $(F_h)_{h\in\mathbb{R}}$ and $F$ be measures on $\mathbb{R}$, such that $F_h(\mathbb{R}) \leq 1$ and $F(\mathbb{R}) \leq 1$. Then the following are equivalent,*

1. $F_h \xrightarrow{\text{weak}} F$ *for* $h \to 0$,
2. *for all continuous bounded functions* $f$,

$$\int f(x)F_h(\mathrm{d}x) \to \int f(x)F(\mathrm{d}x) \quad \text{for} \quad h \to 0,$$

3. *for all* $x$ *where* $F(x)$ *is continuous,* $F_h(x) \to F(x)$ *for* $h \to 0$.

We are now ready to prove the result.

*Proof of Theorem 3.1.* First, see that the boundary condition is evident from inspection of (2.1). Second, we establish differentiability. By Kolmogorov's backwards differential equation (Proposition 2.4) and Theorem 9.2 in [1], we have that

$$t \to p_{ij}\left(s, t, u, d+t-s\right)$$

is continuously differentiable.

Now, let $h > 0$, and consider the Taylor expansion of $p_{ij}\left(s,t,u,z\right)$ around $(s,u)=(t,v)$,

$$\begin{aligned}
&p_{ij}\left(t-h, t, v-h, z\right) \\
&= p_{ij}\left(t,t,v,z\right) - h\,\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}\left(s,t,d+s,z\right)\Big|_{\substack{s=t\\ d+s=v}} + \mathrm{o}(h) \\
&= p_{ij}\left(t,t,v,z\right) - p_{ij}\left(t,t,v,z\right)\mu_{i\cdot}(t,v)h + \sum_{\substack{k\in\mathcal{J}\\ k\neq i}} p_{kj}\left(t,t,0,z\right)\mu_{ik}(t,v)h + \mathrm{o}(h) \\
&= 1_{\{i=j\}}1_{\{v\leq z\}} - 1_{\{i=j\}}1_{\{v\leq z\}}\mu_{i\cdot}(t,v)h + 1_{\{i\neq j\}}\mu_{ij}(t,v)h + \mathrm{o}(h),
\end{aligned}$$

where, in the second equation, we applied Kolmogorov's backwards differential equation, Proposition 2.4.

Before we put it all together, let $(t_0,u)$ and $t$ be fixed, and define

$$F_{ij}(z,h) = p_{ij}\left(t_0, t+h, u, z+h\right),$$

and note that $F_{ij}(\mathrm{d}z, h)$ is a measure, and the support is a subset of $[0, u+t-t_0]$. Since the function $F_{ij}(z, h)$ is continuous in $h$, we have by Lemma A.2 that $F_{ij}(\mathrm{d}z, h) \xrightarrow{\text{weak}} F_{ij}(\mathrm{d}z, 0)$ for $h \to 0$.

Putting it all together, we insert into the Chapman-Kolmogorov equations (A.1),

$$
\begin{aligned}
&F_{ij}(z,0)\\
&= \sum_{\ell\in\mathcal{J}} \int_0^{u+t-t_0} F_{i\ell}(\mathrm{d}v, -h) p_{\ell j}(t-h, t, v-h, z)\\
&= \sum_{\ell\in\mathcal{J}} \int_0^{u+t-t_0} F_{i\ell}(\mathrm{d}v, -h)\Big(1_{\{\ell=j\}}1_{\{v\le z\}}\\
&\qquad - 1_{\{\ell=j\}}1_{\{v\le z\}}\mu_{\ell\cdot}(t,v)h + 1_{\{\ell\neq j\}}\mu_{\ell j}(t,v)h + \mathrm{o}(h)\Big)\\
&= F_{ij}(z,-h) - h\int_0^z F_{ij}(\mathrm{d}v,-h)\mu_{j\cdot}(t,v) + h\sum_{\substack{\ell\in\mathcal{J}\\ \ell\neq j}}\int_0^{u+t-t_0} F_{i\ell}(\mathrm{d}v,-h)\mu_{\ell j}(t,v) + \mathrm{o}(h)
\end{aligned}
$$

and rearrange to obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}h}F_{ij}(z,0)\\
&= \lim_{h\searrow 0}\frac{F_{ij}(z,0)-F_{ij}(z,-h)}{h}\\
&= \lim_{h\searrow 0}\left(-\int_0^z F_{ij}(\mathrm{d}v,-h)\mu_{j\cdot}(t,v) + \sum_{\substack{\ell\in\mathcal{J}\\ \ell\neq j}}\int_0^{u+t-t_0} F_{i\ell}(\mathrm{d}v,-h)\mu_{\ell j}(t,v) + \frac{1}{h}\,\mathrm{o}(h)\right)\\
&= -\int_0^z F_{ij}(\mathrm{d}v,0)\mu_{j\cdot}(t,v) + \sum_{\substack{\ell\in\mathcal{J}\\ \ell\neq j}}\int_0^{u+t-t_0} F_{i\ell}(\mathrm{d}v,0)\mu_{\ell j}(t,v),
\end{aligned}
$$

where we used Lemma A.2 for the last equality. □