Dependent Interest and Transition Rates in Life Insurance

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December 20, 2013

Abstract

For market consistent life insurance liabilities modelled with a multi-state Markov chain, it is of importance to consider the interest and transition rates as stochastic processes, for example in order to consider hedging possibilities of the risks, and for risk measurement. In the literature, this is usually done with an assumption of independence between the interest and transition rates. In this paper, it is shown how to valuate life insurance liabilities using affine processes for modelling dependent interest and transition rates. This approach leads to the introduction of so-called dependent forward rates. We propose a specific model for surrender modelling, and within this model the dependent forward rates are calculated, and the market value and the Solvency II capital requirement are examined for a simple savings contract.

Keywords: Affine Processes; Doubly Stochastic Process; Multi-state Life Insurance Models; Policyholder Behaviour; Solvency II; Surrender

JEL Classification: G22

1 Introduction

Life insurance liabilities are traditionally modelled by a finite state Markov chain with deterministic interest and transition rates. In order to give a market consistent best estimate of the present value of future payments, it has become of increasing interest to let the interest and transition rates be modelled as stochastic processes. The stochastic modelling is important in order to consider hedging possibilities of the risks. With the Solvency II rules, stochastic modelling of the interest and transition rates is also important from a risk management perspective. Modelling the interest and transition

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rates as stochastic processes is traditionally done with an independence assumption. In this paper, we relax the independence assumption, and consider basic valuation with dependence between the interest and one or more transition rates. This is done with continuous affine processes for the modelling of the dependent rates. The study of valuation of life insurance liabilities with dependent rates leads to the definition of so-called dependent forward rates. These are natural quantities that appear in case of dependence, replacing the usual forward rates, which are not directly applicable. Using the theory of dependent affine rates, we consider the case of surrender modelling, and propose a specific model for dependent interest and surrender rates. This is of particular interest from a Solvency II point of view. Within this model, a simple savings contract with a buy-back option is considered. We calculate the dependent forward rates, the market value and the Solvency II capital requirement. This is done in part without hedging, and in part with a simple static hedging strategy. We then examine the effect of correlation between the interest and surrender rate.

The study of valuation of life insurance liabilities with stochastic interest and transition rates has received considerable attention during the last decades. Primarily the interest and mortality rates have been modelled as stochastic, which is often done with affine processes. For basic applications of affine processes for valuation of life insurance liabilities, see [1]. Possibilities of hedging can be considered, which is important for market consistent valuation, and for the study of valuation and hedging of life insurance liabilities with stochastic interest and mortality rates, see [7] and [6]. Another approach to modelling stochastic interest and mortality is taken in [15], where the interest and mortality is modelled within a finite state Markov chain setup. In this paper we extend the study of affine interest and transition rates to the case of dependence. We consider how to valuate life insurance liabilities when the interest and one or more transition rates are modelled as dependent affine processes. This is possible in any decrement/hierarchical Markov chain setup, that is, in Markov chains where, when the process leaves a state, it cannot return. We adopt the theory presented in [4], which is reviewed in Section 2 of this paper. This provides the foundation for the study of multidimensional affine processes in life insurance mathematics. The theory presented in [4] is based partly on a result in [8], and partly on general theory for multidimensional affine processes presented in [9].

In the financial literature, the concept of a forward interest rate exists, which is convenient, e.g. for representing zero coupon bond prices. This quantity appears naturally in life insurance mathematics, when the interest rate is modelled as a stochastic process. If one also considers a stochastic mortality, independent of the interest rate, it becomes natural to define a forward mortality rate as well. With these forward rates, the expected present value of the life insurance liabilities has a particularly compelling representation. However, if one introduces dependence between the interest and mortality rates, the forward rates are no longer applicable. In this paper, we introduce so-called dependent forward rates that appear naturally and are applicable for representing the expected present value of the life insurance liabilities in a convenient form, in cases where the usual forward rates are not applicable. In [11], alternative forward mortality rates are defined in order to handle the case of dependence. In the present paper, we show that one of the forward mortality rates defined in [11] is in general not well defined. For a general discussion on forward rates, and their usefulness, see [14], wherein the case of dependence between the rates is discussed as well. One of the consistency problems with forward rates in the dependent setup that is raised in [14] is solved by the proposed dependent forward rates introduced in the present paper. Also, the dependent forward rates introduced here generalise the usual definitions of forward rates, in the sense that when there is independence between the rates, the dependent forward rates equal the usual forward rates.

Modelling policyholder behaviour has become of increasing importance with the proposed Solvency II rules, where one is required to consider any dependence between the economic environment and policyholder behaviour, see Section 3.5 in [5]. The study of surrender behaviour can either be made using a rational approach, where the outset is, that the policyholders surrender the contract if it is rational from some economic viewpoint, which is studied in [16]. This seems a bit extreme, given that this behaviour is not seen in practice. Another approach is the intensity approach, where the policyholders surrender randomly, regardless whether or not it is profitable in the current economic environment. This is not a perfect way of modelling either, since if the interest rates decrease a lot, a guarantee given in connection with the life insurance contract motivates the policyholders to keeping the contract. For an overview of some of the approaches, see [12]. In [10], an attempt is made on coupling the two approaches, using two different surrender rate models if it is rational or irrational, respectively, to surrender. In this paper, we propose another way of coupling the two approaches. We let the surrender rate be positively correlated with the interest rate, thus if the interest rate decreases a lot, the surrender rate also decreases, representing that the guarantee inherent in the life insurance contract is of value to the policyholder.

The Solvency II capital requirement is basically, that "the insurance company must have enough capital, such that the probability of default within the next year is less than 0.5%", representing that a default is a 200-year event. When the insurance company updates its mortality tables, or other transition rate tables, this represents a risk that must be taken into account when putting up the Solvency II capital requirement. Mathematically, this can be done using stochastic rates. For an examination of mortality modelling and the Solvency II capital requirement, see e.g. [2]. For a basic discussion of the mathematical formulation of the Solvency II capital requirement, see e.g. [3]. In this paper, we determine the Solvency II capital requirement for the simple savings contract where the interest and surrender rate risk is considered, both in the case of no hedging strategy, and also in the case of a simple strategy where interest rate risk is hedged.

The structure of the paper is as follows. In Section 2, we present basic results on multidimensional continuous affine processes, which provides the foundation for the application of dependent affine processes in life insurance mathematics. In Section 3, we present the general life insurance setup with stochastic interest and transition rates, and in Section 4, we propose the definition of dependent forward rates and compare to the usual forward rate definition. In Section 4.1, we discuss other definitions in the literature of forward rates in a dependent setup, and compare them to the dependent forward rates proposed here. In Section 5, we present a specific model for dependent interest and surrender rates. The model is introduced in Section 5.1. We first discuss how to find the Solvency II capital requirement, which is done in Section 5.3, and a simple hedging strategy for the interest rate risk is presented in Section 5.4. Numerical results are presented in Section 5.5, consisting of the dependent forward rates found, and the market value and Solvency II capital requirement, presented for different levels of correlation.

2 Continuous Affine Processes

The class of affine processes provides a method for modelling interest and transition rates, with the possibility of adding dependence. In this section, we consider general results about continuous affine processes, which we apply in this paper. For more details on the theory presented in this section, see [4].

Let $\mathbf{X}$ be a d-dimensional affine process, satisfying the stochastic differential equation

$$\mathrm{d}X(t) = (b(t) + \mathcal{B}(t)X(t))\,\mathrm{d}t + \rho(t, X(t))\,\mathrm{d}W(t),$$

where $\mathbf{W}$ is a d-dimensional Brownian motion. Here, $b : \mathbb{R}_+ \to \mathbb{R}^d$ is a vector function, and $\mathcal{B} : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ is a matrix function, where we denote column i by $\beta_i(t)$, so that $\mathcal{B}(t) = (\beta_1(t), \dots, \beta_d(t))$. When squared, the volatility parameter function $\rho(t, x)$ must be affine in x, i.e.

$$\rho(t, x)\rho(t, x)^\top = a(t) + \sum_{i=1}^{d} \alpha_i(t)x_i,$$

for matrix functions $a : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ and $\alpha_i : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$. Consider now affine transformations of $\mathbf{X}$, by defining a vector function $c : \mathbb{R}_+ \to \mathbb{R}^p$ and a matrix function $\Gamma : \mathbb{R}_+ \to \mathbb{R}^{p\times d}$, thereby defining the p-dimensional process,

$$Y(t) = c(t) + \Gamma(t)X(t). \tag{2.1}$$

We think of $\mathbf{X}$ as socio-economic driving factors, and then $\mathbf{Y}$ is a collection of the stochastic interest rate and/or transition rates. In this section, we work in a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}(t))_{t\in\mathbb{R}_+}$ generated by the Brownian motion $\mathbf{W}$.

For applications of $\mathbf{Y}$ as interest and as transition rates in finite state Markov chain models, we present some essential relations. The results hold under certain regularity conditions, for details see [4]. Denote by $\mathbb{1}$ a vector with 1 in all entries, where the dimension is implicit. Also, denote by $\gamma_i(t)$ the sum of the ith column in $\Gamma(t)$, i.e. $\gamma_i(t) = \mathbb{1}^\top \Gamma(t) e_i$, where e_i is the ith unit vector, $i = 1, \dots, d$.

The first relation, the basic pricing formula, is for $0 \le t \le T$ given by

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} \middle| \mathcal{F}(t) \right] = e^{\phi(t,T)+\psi(t,T)^\top X(t)}, \tag{2.2}$$

where $\phi(t,T)$ is a real function, and $\psi(t,T)$ is a d-dimensional function, given by the system of differential equations,

$$\begin{aligned} \frac{\partial}{\partial t}\phi(t,T) &= -\frac{1}{2}\psi(t,T)^\top a(t)\psi(t,T) - b(t)^\top \psi(t,T) + \mathbb{1}^\top c(t), \\ \frac{\partial}{\partial t}\psi_i(t,T) &= -\frac{1}{2}\psi(t,T)^\top \alpha_i(t)\psi(t,T) - \beta_i(t)^\top \psi(t,T) + \gamma_i(t), \quad i = 1, \dots, d, \end{aligned} \tag{2.3}$$

with boundary conditions $\phi(T,T) = 0$ and $\psi(T,T) = 0$.

For the second relation, let a vector $\kappa \in \mathbb{R}^p$ be given, and let $u \in [t,T]$ be some time point. Then,

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} \kappa^\top Y(u) \middle| \mathcal{F}(t) \right] = e^{\phi(t,T)+\psi(t,T)^\top X(t)} \Big(A(t,T,u) + B(t,T,u)^\top X(t) \Big), \tag{2.4}$$

where (ϕ, ψ) is given by (2.3) as above, A is a real function and B is a vector function, given by the system of differential equations,

$$\begin{aligned} \frac{\partial}{\partial t}A(t,T,u) &= -\psi(t,T)^\top a(t)B(t,T,u) - b(t)^\top B(t,T,u), \\ \frac{\partial}{\partial t}B_i(t,T,u) &= -\psi(t,T)^\top \alpha_i(t)B(t,T,u) - \beta_i(t)^\top B(t,T,u), \quad i = 1, \dots, d, \end{aligned} \tag{2.5}$$

with boundary conditions $A(u,T,u) = \kappa^\top c(u)$ and $B(u,T,u) = \kappa^\top \Gamma(u)$. A particular example of importance is $\kappa = e_k$ for some $k = 1, \dots, p$, and in this case, we write A^k and B^k to emphasize the dependence on k. This second relation (2.4) is proven in [8] for $u = T$, and the extension to the case $u < T$ is for example given in [4].

The third relation is, for another time point $v \in [t, T]$, and two integers $k, l = 1, \dots, p$, given by

$$
\begin{aligned}
\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} Y_k(u) Y_l(v) \,\middle|\, \mathcal{F}(t) \right] &= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \\
\times \Big\{ \Big(A^k(t,T,u) + B^k(t,T,u)^\top X(t) \Big) &\Big(A^l(t,T,v) + B^l(t,T,v)^\top X(t) \Big) \\
+ C^{kl}(t,T,u,v) + D^{kl}(t,T,u,v)^\top X(t) \Big\},
\end{aligned} \tag{2.6}
$$

where (ϕ, ψ) solves (2.3) and (A^k, B^k) and (A^l, B^l) both solve (2.5) with boundary conditions $A^k(u,T,u) = e_k^\top c(u)$, $B^k(u,T,u) = e_k^\top \Gamma(u)$ and $A^l(v,T,v) = e_l^\top c(v)$, $B^l(v,T,v) = e_l^\top \Gamma(v)$, respectively. The functions C^{kl} and D^{kl} are determined by the following system of differential equations,

$$
\begin{aligned}
\frac{\partial}{\partial t} C^{kl}(t,T,u,v) &= -B^k(t,T,u)^\top a(t) B^l(t,T,v) \\
&\quad - \psi(t,T)^\top a(t) D^{kl}(t,T,u,v) - b(t)^\top D^{kl}(t,T,u,v), \\
\frac{\partial}{\partial t} D_i^{kl}(t,T,u,v) &= -B^k(t,T,u)^\top \alpha_i(t) B^l(t,T,v) \\
&\quad - \psi(t,T)^\top \alpha_i(t) D^{kl}(t,T,u,v) - \beta_i(t)^\top D^{kl}(t,T,u,v),
\end{aligned} \tag{2.7}
$$

for $i = 1, \dots, d$, with boundary conditions[1] $C^{kl}(u \wedge v, T, u, v) = 0$ and $D^{kl}(u \wedge v, T, u, v) = 0$. This result is proven in [4].

3 The Life Insurance Model

Consider the usual life insurance setup. Let $\mathbf{Z} = (Z(t))_{t \in \mathbb{R}_+}$ be a Markov process in the finite state space $\mathcal{J}$, indicating the state of the insured. The distribution of $\mathbf{Z}$ is defined via the transition rates $(\mu_{ij}(t))_{t \in \mathbb{R}_+}$, $i, j \in \mathcal{J}$. With $(N_{ij}(t))_{t \in \mathbb{R}_+}$, $i, j \in \mathcal{J}$ being the process that counts the number of jumps for $\mathbf{Z}$ from state i to j, the compensated process

$$
N_{ij}(t) - \int_0^t 1_{(Z(s-)=i)} \mu_{ij}(s)\,\mathrm{d}s
$$

is a martingale. We can allow the transition rates (μ_{ij}) to be stochastic. In this case, the distribution of $\mathbf{Z}$ is defined conditionally on the transition rates.

We model the transition rates as a time-dependent affine transformation of a d-dimensional continuous affine process $\mathbf{X}$. That is, for functions $c : \mathbb{R}_+ \to \mathbb{R}^p$ and $\Gamma : \mathbb{R}_+ \to \mathbb{R}^{p \times d}$, let

[1] The notation $x \wedge y = \min\{x, y\}$ is used.

$\mathbf{Y}$ be defined as

$$Y(t) = c(t) + \Gamma(t)X(t).$$

Hence, each of the stochastic transition rates are modelled as an element in $\mathbf{Y}$.

The interest rate process $(r(t))_{t\in\mathbb{R}_+}$ is also allowed to be stochastic. This is modelled in the same way, by specifying r as an element in $\mathbf{Y}$. By the design of Γ and $\mathbf{X}$, the interest and transition rates can be dependent, independent or deterministic.

Let the filtrations $\mathbb{F}^{\mathbf{Z}} = (\mathcal{F}^{\mathbf{Z}}(t))_{t\in\mathbb{R}_+}$ and $\mathbb{F}^{\mathbf{X}} = (\mathcal{F}^{\mathbf{X}}(t))_{t\in\mathbb{R}_+}$ be the ones generated by the processes $\mathbf{Z}$ and $\mathbf{X}$, respectively, satisfying the usual hypothesis. We consider the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}(t))_{t\in\mathbb{R}_+}$ is given by $\mathcal{F}(t) = \mathcal{F}^{\mathbf{Z}}(t) \vee \mathcal{F}^{\mathbf{X}}(t)$.

We consider a life insurance policy, with payments specified by the process $\mathbf{B} = (B(t))_{t\in\mathbb{R}_+}$, such that $B(t)$ is the total payments until time t. Then we can think of $\mathrm{d}B(t)$ as the payment at time t, and we can specify $\mathbf{B}$ as

$$\mathrm{d}B(t) = \sum_{i\in\mathcal{J}} 1_{(Z(t)=i)} b_i(t)\,\mathrm{d}t + \sum_{\substack{i,j\in\mathcal{J}\\ i\neq j}} b_{ij}(t)\,\mathrm{d}N_{ij}(t),$$

for deterministic payment functions b_i and b_{ij}, $i, j \in \mathcal{J}$. Then $b_i(t)$ is the payment while in state i at time t, and $b_{ij}(t)$ is the payment if jumping from state i to j at time t.

The present value at time t of the future payments associated with the life insurance policy is given by

$$PV(t) = \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}B(s).$$

For reserving and pricing, one considers the expected present value

$$V(t) = \mathrm{E}\left[\int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}B(s)\middle|\,\mathcal{F}(t)\right],$$

where the expectation is taken using a market, risk neutral or pricing measure. For actually calculating $V(t)$, the tower property is applied, that is, we condition on $\mathcal{F}^{\mathbf{X}}(\infty)$ to get

$$V^{\mathbf{X}}(t) = \mathrm{E}\left[\int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}B(s)\middle|\,\mathcal{F}^{\mathbf{Z}}(t) \vee \mathcal{F}^{\mathbf{X}}(\infty)\right],$$

so that $V(t) = \mathrm{E}\left[V^{\mathbf{X}}(t)\middle|\,\mathcal{F}(t)\right]$. Here, $V^{\mathbf{X}}(t)$ is the reserve conditional on the interest and transition rates, thus corresponding to the case of deterministic rates. When valuating $V^{\mathbf{X}}$ we need the conditional distribution of $\mathbf{Z}$, and thus $\mathbf{B}$, given the transition rates. By construction this is known, and well-established theory about life insurance reserves with deterministic interest and transition rates (see e.g. [13]) hold.

Example 3.1. Consider a surrender model with 3 states $\mathcal{J} = \{0, 1, 2\}$, corresponding to *alive*, *dead* and *surrendered* respectively. The Markov model is shown in Figure 1. Let the transition rate from state *alive* to state *dead*, i.e. the mortality rate, be deterministic. We model the interest rate r and the surrender rate η as dependent affine processes in the form,

$$(r(t), \eta(t))^\top = c(t) + \Gamma(t)X(t),$$

for a d-dimensional affine process $\mathbf{X}$. Hence, this specification is analog to (2.1). By the design of $\mathbf{X}$, the processes $\mathbf{X}_i$, $i = 1, \dots, d$ can be dependent processes, such that the interest rate r and the surrender rate η can be dependent processes.

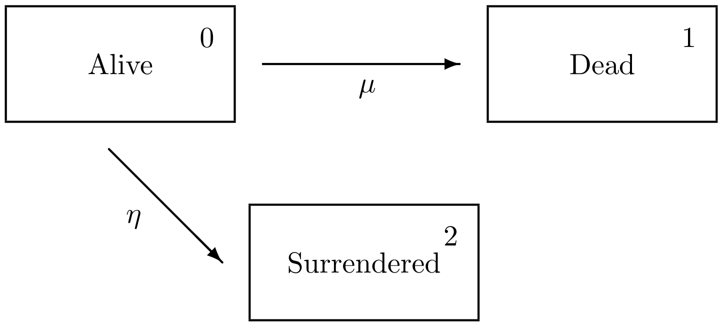

Figure 1: *Markov model for the survival-surrender model.*

Let the payments be defined by

$$\mathrm{d}B(t) = b(t)1_{(Z(t)=0)}\,\mathrm{d}t + b_d(t)\,\mathrm{d}N_{01}(t) + U(t)\,\mathrm{d}N_{02}(t),$$

where $b(t)$ is the continuous payment rate at time t while alive, $b_d(t)$ is the single payment if death occurs at time t, and $U(t)$ is the payment upon surrender at time t. The payment functions are deterministic.

Conditioning on the intensitites, the expected present value $V^{\mathbf{X}}(t)$ is the classic result,

$$\begin{aligned} V^{\mathbf{X}}(t) &= \mathrm{E}\left[PV(t) \mid \mathcal{F}^{\mathbf{X}}(\infty), Z(t) = 0\right] \\ &= \int_t^\infty e^{-\int_t^s (r(\tau)+\mu(\tau)+\eta(\tau))\,\mathrm{d}\tau} \left(b(s) + \mu(s)b_d(s) + \eta(s)U(s)\right)\,\mathrm{d}s, \end{aligned}$$

see e.g. [13]. Removing the condition, we find, using Equations (2.2) and (2.4),

$$V(t) = \mathrm{E}\left[V^{\mathbf{X}}(t) \middle| \mathcal{F}(t)\right]$$

$$
\begin{aligned}
&= \int_t^\infty e^{-\int_t^s \mu(\tau)\,\mathrm{d}\tau} \bigg\{ \mathrm{E}\left[e^{-\int_t^s (r(\tau)+\eta(\tau))\,\mathrm{d}\tau} \,\Big|\, \mathcal{F}(t)\right] (b(s) + \mu(s) b_d(s)) \\
&\qquad + \mathrm{E}\left[e^{-\int_t^s (r(\tau)+\eta(\tau))\,\mathrm{d}\tau} \eta(s) \,\Big|\, \mathcal{F}(t)\right] U(s) \bigg\}\,\mathrm{d}s \\
&= \int_t^\infty e^{-\int_t^s \mu(\tau)\,\mathrm{d}\tau + \phi(t,s) + \psi(t,s)^\top X(t)} \Big(b(s) + \mu(s) b_d(s) \\
&\qquad + \Big(A^\eta(t,s,s) + B^\eta(t,s,s)^\top X(t) \Big) U(s) \Big)\,\mathrm{d}s.
\end{aligned}
$$

○

4 Dependent Forward Rates

The form of $V(t)$ in Example 3.1 motivates the definition of quantities similar to forward rates, that can be used to express the solution. In particular, this leads to a forward interest rate, but this is in general not equal the forward rate obtained using the usual definition. Hence, we apply the term *dependent forward rates*.

Let $\mathbf{X}$, $c(t)$ and $\Gamma(t)$ be given, and let $\mathbf{Y}$ be of the form (2.1). We consider some motivating calculations first, and then define the dependent forward rates. See that,

$$
\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} \mathbb{1}^\top Y(T) \,\Big|\, \mathcal{F}(t)\right] \\
&= -\frac{\partial}{\partial T} \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} \,\Big|\, \mathcal{F}(t)\right] \\
&= -\frac{\partial}{\partial T} e^{\phi(t,T) + \psi(t,T)^\top X(t)} \\
&= e^{\phi(t,T) + \psi(t,T)^\top X(t)} \left(-\frac{\partial}{\partial T}\phi(t,T) + X(t)^\top \left(-\frac{\partial}{\partial T}\psi(t,T) \right) \right),
\end{aligned}
$$

where we interchanged integration and differentiation, and applied (2.2). On the other hand, if we instead apply (2.4) with $\kappa = \mathbb{1}$, we find

$$
\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} \mathbb{1}^\top Y(T) \,\Big|\, \mathcal{F}(t)\right] \\
&= e^{\phi(t,T) + \psi(t,T)^\top X(t)} \Big(A(t,T,T) + X(t)^\top B(t,T,T) \Big), \\
&= e^{\phi(t,T) + \psi(t,T)^\top X(t)} \left(\sum_{k=1}^p A^k(t,T,T) + X(t)^\top \sum_{k=1}^p B^k(t,T,T) \right),
\end{aligned}
$$

where (A^k, B^k), $k = 1, \ldots, p$ are solutions to (2.5) with boundary conditions given by $\kappa = e_k$, i.e. $A^k(T,T,T) = e_k^\top c(T)$ and $B^k(T,T,T) = e_k^\top \Gamma(T)$. The last equality sign is obtained using the relations $\sum_{k=1}^p A^k(t,T,T) = A(t,T,T)$ and $\sum_{k=1}^p B^k(t,T,T) = B(t,T,T)$, which hold since (A, B) also solves the linear system of differential equations

(2.5), with boundary conditions given by $\kappa = \mathbb{1}$. Gathering the two calculations above, we conclude that

$$-\frac{\partial}{\partial T}\phi(t,T) = \sum_{k=1}^{p} A^k(t,T,T), \qquad -\frac{\partial}{\partial T}\psi(t,T) = \sum_{k=1}^{p} B^k(t,T,T),$$

and, in particular, since $\phi(t,t) = 0$ and $\psi(t,t) = 0$, that

$$\phi(t,T) = -\int_t^T \sum_{k=1}^{p} A^k(t,s,s)\,\mathrm{d}s, \qquad \psi(t,T) = -\int_t^T \sum_{k=1}^{p} B^k(t,s,s)\,\mathrm{d}s. \tag{4.1}$$

Definition 4.1. *Let* $\mathbf{X}$ *be a* d*-dimensional continuous affine process, and let* c *and* Γ *be given, such that* $\mathbf{Y}$ *from* (2.1) *is defined. Let* $t \leq s$ *and* $k = \{1, \ldots, p\}$*. The* dependent forward rate $f_t^k(s)$ *for the stochastic rate* $Y_k(s)$ *at time* t *is then given by*

$$f_t^k(s) = A^k(t,s,s) + X(t)^\top B^k(t,s,s), \tag{4.2}$$

where (A^k, B^k) *solves the system of differential equations* (2.5)*, with boundary conditions given by* $\kappa = e_k$*.*

Remark 4.2. Using the notation of the dependent forward rates, we can express the relation (2.2), and for $u = T$ also the relation (2.4), as

$$\begin{aligned} \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s}\middle|\,\mathcal{F}(t)\right] &= e^{-\int_t^T \sum_{i=1}^{p} f_t^i(s)\,\mathrm{d}s}, \\ \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} Y_k(T)\middle|\,\mathcal{F}(t)\right] &= e^{-\int_t^T \sum_{i=1}^{p} f_t^i(s)\,\mathrm{d}s} f_t^k(T). \end{aligned} \tag{4.3}$$

◇

The dependent forward rates are in Definition 4.1 only for affine processes. However, if one wish, equation (4.3) can be used to extend the definition to any underlying process: The equations (4.3) uniquely determine the dependent forward rates, thus the dependent forward rates exist for any underlying process, and not only affine processes. This is not a constructive definition though, and in the present paper we only focus on the affine class.

Example 4.3. (Example 3.1 continued) Using the definition of the dependent forward rates, we can write the expected present value as,

$$V(t) = \int_t^\infty e^{-\int_t^s \left(f_t^r(\tau) + \mu(\tau) + f_t^\eta(\tau)\right)\mathrm{d}\tau}\Big(b(s) + \mu(s)b_d(s) + f_t^\eta(s)U(s)\Big)\,\mathrm{d}s. \tag{4.4}$$

We see that the expected present value is of the same form as the formula that appears in the case of deterministic rates, but with the interest and surrender rates exchanged by the corresponding dependent forward rates. Note that we used a slightly different notation, such that we write f^r instead of f^1 and f^η instead of f^2.

Often we want to consider both the quantity

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s}\middle|\,\mathcal{F}(t)\right] = \mathrm{E}\left[e^{-\int_t^T (r(s)+\eta(s))\,\mathrm{d}s}\middle|\,\mathcal{F}(t)\right],$$

where $Y(t) = (r(t), \eta(t))$, as well as the quantities arising from the models $Y^1(t) = (r(t), 0)$ and $Y^2(t) = (0, \eta(t))$,

$$\begin{aligned}\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y^1(s)\,\mathrm{d}s}\middle|\,\mathcal{F}(t)\right] &= \mathrm{E}\left[e^{-\int_t^T r(s)\,\mathrm{d}s}\middle|\,\mathcal{F}(t)\right],\\ \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y^2(s)\,\mathrm{d}s}\middle|\,\mathcal{F}(t)\right] &= \mathrm{E}\left[e^{-\int_t^T \eta(s)\,\mathrm{d}s}\middle|\,\mathcal{F}(t)\right].\end{aligned}$$

In such cases, we add a more detailed superscript to the forward rates f, and specify the model we think of after a colon. That is, we write

$$\mathrm{E}\left[e^{-\int_t^T (r(s)+\eta(s))\,\mathrm{d}s}\middle|\,\mathcal{F}(t)\right] = e^{-\int_t^T (f_t^{r:(r+\eta)}(s)+f_t^{\eta:(r+\eta)}(s))\,\mathrm{d}s},$$

as well as

$$\begin{aligned}\mathrm{E}\left[e^{-\int_t^T r(s)\,\mathrm{d}s}\middle|\,\mathcal{F}(t)\right] &= e^{-\int_t^T f_t^{r:r}(s)\,\mathrm{d}s},\\ \mathrm{E}\left[e^{-\int_t^T \eta(s)\,\mathrm{d}s}\middle|\,\mathcal{F}(t)\right] &= e^{-\int_t^T f_t^{\eta:\eta}(s)\,\mathrm{d}s}.\end{aligned}$$

Note that $f_t^{r:r}(s)$ and $f_t^{\mu:\mu}(s)$ are the usual forward rates. ○

The representation (4.4) is convenient, since it allows us to use the classic formulae, and just plug in pre-calculated dependent forward rates. The result is only obtainable with the dependent forward rates defined here. If one used a spread-rate approach, as in [11], one would have had two different surrender rates and thus not obtained the formula (4.4). In [14], Section 5, the forward rate approach is criticised by the fact that the formula (4.4) is not available, and the dependent forward rates meet this critique. The difference between the dependent forward rates and the spread rate approach is examined with (4.11) in Section 4.1 below.

We briefly compare with the usual forward interest rate. Let the model $Y(t) = c(t) + \Gamma(t)X(t)$ be given, for $p > 1$, and let $r(t) = Y_1(t)$ be the interest rate. The *forward interest rate* is the function $g_t(s)$ that satisfies

$$\mathrm{E}\left[e^{-\int_t^T r(s)\,\mathrm{d}s}\middle|\,\mathcal{F}(t)\right] = e^{-\int_t^T g_t(s)\,\mathrm{d}s}.$$

This function also satisfies, as can be shown by differentiation,

$$\mathrm{E}\left[e^{-\int_t^T r(s)\,\mathrm{d}s} r(T)\middle|\,\mathcal{F}(t)\right] = e^{-\int_t^T g_t(s)\,\mathrm{d}s} g_t(T). \tag{4.5}$$

The dependent forward rate for the interest rate in our model $\mathbf{Y}$, as defined in Definition 4.1, is denoted $f_t^r(s)$. It satisfies,

$$\mathrm{E}\left[e^{-\int_t^T (r(s)+Y_2(s)+\ldots+Y_p(s))\,\mathrm{d}s} r(T) \middle| \mathcal{F}(t)\right] = e^{-\int_t^T \left(f_t^r(s)+f_t^2(s)+\ldots+f_t^p(s)\right)\mathrm{d}s} f_t^r(T), \quad (4.6)$$

where the other forward rates $f_t^i(s)$ satisfy analogue relations.

In the case that $\mathbf{r} = (r(t))_{t\in\mathbb{R}_+}$ is independent of $\mathbf{Y}_2, \ldots, \mathbf{Y}_p$, the dependent forward rate for the interest $\mathbf{r}$ simplifies to the usual forward interest rate. This can be seen by two simple calculations. First, see that

$$\begin{aligned}
&e^{-\int_t^T \left(f_t^r(s)+f_t^2(s)+\ldots+f_t^p(s)\right)\mathrm{d}s} \\
&= \mathrm{E}\left[e^{-\int_t^T (r(s)+Y_2(s)+\ldots+Y_p(s))\,\mathrm{d}s} \middle| \mathcal{F}(t)\right] \\
&= \mathrm{E}\left[e^{-\int_t^T r(s)\,\mathrm{d}s} \middle| \mathcal{F}(t)\right] \mathrm{E}\left[e^{-\int_t^T (Y_2(s)+\ldots+Y_p(s))\,\mathrm{d}s} \middle| \mathcal{F}(t)\right] \\
&= e^{-\int_t^T g_t(s)\,\mathrm{d}s}\, \mathrm{E}\left[e^{-\int_t^T (Y_2(s)+\ldots+Y_p(s))\,\mathrm{d}s} \middle| \mathcal{F}(t)\right].
\end{aligned} \quad (4.7)$$

A similar calculation, using (4.6) and (4.5), yields

$$\begin{aligned}
&e^{-\int_t^T \left(f_t^r(s)+f_t^2(s)+\ldots+f_t^p(s)\right)\mathrm{d}s} f_t^r(T) \\
&\quad = e^{-\int_t^T g_t(s)\,\mathrm{d}s} g_t(T)\, \mathrm{E}\left[e^{-\int_t^T (Y_2(s)+\ldots+Y_p(s))\,\mathrm{d}s} \middle| \mathcal{F}(t)\right].
\end{aligned}$$

Dividing with the identity (4.7) above, we conclude that

$$f_t^r(T) = g_t(T),$$

which holds for all T where $t < T$, and we conclude that the dependent forward interest rate equals the usual forward interest rate.

The calculations relied critically on the independence assumption, and in the general case the dependent forward rate for the interest is not equal to the forward interest rate. Intuitively, when the interest rate appears together with other dependent rates, the forward rates need to compensate for this dependence, and thus the difference of the dependent forward rates and the usual forward rates can be thought of as a "covariance" term.

4.1 Comparison With Other Dependent Setups

For the case of dependent affine rates, there have been other proposals for the definition of forward rates. In [11], the model contains an interest rate and a mortality rate which are dependent. This corresponds to the case $p = 2$, where $r(t) = Y_1(t)$ is the interest rate and $\mu(t) = Y_2(t)$ is the mortality rate. Their approach is to keep the definition of the forward interest rate $g_t : [t, \infty) \to \mathbb{R}_+$ unchanged, and then find forward mortality rates that are compatible with this definition, thus interpreting the forward mortality rate as a spread rate. In order to make this idea work, they define two different mortality rates, one for pure endowments, and one for term insurances. We briefly review this approach and compare to the definition of the dependent forward rates in the previous section. This serves to highlight the advantage of the dependent forward rates, in particular that there is only one forward mortality rate when considering a life annuity and a term insurance together.

The *forward mortality rate for pure endowments*, $h_t^{\mathrm{pe}} : [t, \infty) \to \mathbb{R}_+$, is defined as the function satisfying

$$\mathrm{E}\left[e^{-\int_t^T (r(s)+\mu(s))\,\mathrm{d}s} \middle| \mathcal{F}(t) \right] = e^{-\int_t^T \left(g_t(s)+h_t^{\mathrm{pe}}(s)\right)\mathrm{d}s}.$$

In terms of the dependent forward rates, f_t^r and f_t^μ, we can use the first part of (4.3) and write the forward mortality rate for pure endowment as,

$$h_t^{\mathrm{pe}}(s) = f_t^r(s) + f_t^\mu(s) - g_t(s), \tag{4.8}$$

which in particular shows that it is well-defined. The forward mortality rate for pure endowment can be given an intuitive interpretation. Recall that the dependent forward rates are different from the usual definition of forward rates, because the mortality rate appears together with another dependent rate, thus the dependent forward rates contains a "covariance" part. The forward mortality rate for pure endowments corresponds to moving the "covariance" from the forward interest rate into the forward mortality rate, instead of having a part in each of the forward rates. In other words, $f_t^r + f_t^\mu$ contains the "covariance" terms, and subtracting g_t, which does not contain any "covariance" terms, the "covariance" terms are contained in h_t^{pe}. In this way, the original definition of the forward interest rate can be kept unaltered, but one can say that the forward mortality rate for pure endowment h_t^{pe} contains a "covariance" term belonging to the interest rate.

The *forward mortality rate for term insurances*, $h_t^{\mathrm{ti}} : [t, \infty) \to \mathbb{R}_+$, is defined as the function satisfying,

$$\mathrm{E}\left[\int_t^T e^{-\int_t^u (r(s)+\mu(s))\,\mathrm{d}s} \mu(u)\,\mathrm{d}u \middle| \mathcal{F}(t) \right] = \int_t^T e^{-\int_t^u \left(g_t(s)+h_t^{\mathrm{ti}}(s)\right)\mathrm{d}s} h_t^{\mathrm{ti}}(u)\,\mathrm{d}u. \tag{4.9}$$

To establish that h_t^{ti} is well-defined is not as easy as with the forward mortality rate for the pure endowment. First, see that the definition depends on the choice of T. It is natural to make the assumption that the forward mortality rate for term insurances h_t^{ti} is independent of T. This assumption is implicity made in the notation used in [11], and the assumption is also made for the forward mortality rate for pure endowment. With this assumption of independence of T, we can differentiate with respect to T, and find the equivalent definition,

$$\mathrm{E}\left[e^{-\int_t^T (r(s)+\mu(s))\,\mathrm{d}s} \mu(T) \middle| \mathcal{F}(t) \right] = e^{-\int_t^T \left(g_t(s)+h_t^{\mathrm{ti}}(s)\right)\mathrm{d}s} h_t^{\mathrm{ti}}(T), \tag{4.10}$$

for $T \geq t$. We are now ready to answer the question of well-definedness, which is important for a fruitful definition of a forward rate. It turns out, that when using the definition (4.10), there are cases where the forward mortality rate for term insurances does not exist for all time points, and one should therefore be careful to use the definition in practice. This will in particular be the case for models with positive correlation. A proof is given in the appendix, where we in Section A present a class of models where the forward mortality rate for term insurances does not exist. If one instead uses the definition (4.9) and allow $h_t^{\mathrm{ti}}(s)$ to depend on T as well, the extra parameter T probably makes it possible to show that it is well-defined.

4.1.1 One forward mortality/surrender rate

We are now ready to present the main difference between the spread rate approach and the dependent forward rate approach. We compare the forward mortality rates from [11] with the dependent forward rates, and for now assuming that the forward rate for term insurances exists, we consider a policy consisting of a life annuity with a payment rate b, and a term insurance with payment 1 upon death. The policy terminates at time T. The expected present value at time t is

$$\begin{aligned}
&\mathrm{E}\left[\int_t^T e^{-\int_t^s (r(s)+\mu(s))\,\mathrm{d}s} \left(b+\mu(s)\right)\,\mathrm{d}s \middle| \mathcal{F}(t) \right] \\
&= \int_t^T e^{-\int_t^s (f_t^{r:(r+\mu)}(s)+f_t^{\mu:(r+\mu)}(s))\,\mathrm{d}s} \left(b+f_t^{\mu:(r+\mu)}(s)\right)\,\mathrm{d}s \\
&= \int_t^T e^{-\int_t^s g_t(s)\,\mathrm{d}s} \left(e^{-\int_t^s h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} b + e^{-\int_t^s h_t^{\mathrm{ti}}(s)\,\mathrm{d}s} h_t^{\mathrm{ti}}(s)\right)\mathrm{d}s,
\end{aligned} \tag{4.11}$$

where we first wrote it in terms of the dependent forward rates, and afterwards in terms of the forward mortality rates for pure endowment and term insurances, respectively. This illustrates the difference between the different types of forward rates. The dependent forward rate for mortality can be used for both the life annuity and the term insurance, whereas with the other forward mortality rate definitions, one need a different one for a different product. If the interest rate is independent of the mortality rate, the different forward mortality rates simplify and they all equal the usual forward mortality rate.

The fact that the dependent forward rates solve the problem of the two different forward rates for the different products in (4.11) is one of the main advantages. It is exactly this problem with existing forward rates that is criticised in Section 5 of [14]. With the dependent forward rates, this issue is resolved in that a unique forward mortality rate exists, that can be used for both the life annuity and the term insurance. In the article [14] a general discussion of the concept of forward rates, and generalisations to dependent models is carried out, including discussion of requirements for more generalised forward rates. Even though the critique from Section 5 of [14] is met with the dependent forward rates, they do not meet all the requirements set up in [14]. In particular, in life insurance models where one needs to use the relation (2.6), the dependent forward rates are not applicable.

The feature of a unique forward mortality rate in (4.11) does also apply to the surrender setup in Example 4.3. The dependent forward rates allow us to have one forward surrender rate, and if the spread approach was used, one would have had different forward surrender rates: One for reducing with the probability of not having surrendered, and another for calculating the probability of surrendering at an exact time. Thus, the formula (4.4) would have had two parts with two different forward surrender rates, similar to the last line in (4.11).

5 Modelling Interest and Surrender

In order to illustrate the methods proposed, we put up a specific model for dependent interest and surrender rate. The results are presented naturally using the dependent forward rates such that the formulae are in parallel with the classic life insurance results obtained with deterministic transition rates. This allows for convenient interpretation and comparison of the results. For example, using the replacement result, the effects of introducing stochastic rates can be measured in terms of forward rates, i.e. the difference between the original deterministic rates and the dependent forward rates.

We model the interest rate as a stochastic diffusion process r, and the surrender rate by the diffusion process η. The interest and surrender rates are then modelled as dependent processes, within the affine setup presented in Section 2. Within the Solvency II regime, one is required to model surrender behaviour, and also take into consideration any dependence of the interest rate (i.e. the economic environment), see Section 3.5 in [5]. This model is thus an example of how this can be done.

5.1 Correlated Interest and Surrender Model

Let $\eta^0(t)$ be a deterministic surrender rate, corresponding to best estimate, i.e. the expectation of the future surrender rate. The interest rate $r(t)$ and surrender rate $\eta(t)$

are then modelled as an affine transformation of $\mathbf{X}$ of the form,

$$\begin{aligned} r(t) &= X_1(t), \\ \eta(t) &= \eta^0(t) X_2(t), \end{aligned}$$

where $\mathbf{X}$ is a 2-dimensional stochastic diffusion process. The process $\mathbf{X}$ satisfies the stochastic differential equation,

$$\begin{aligned} \mathrm{d}X_1(t) &= (b_1(t) - \beta_1 X_1(t))\,\mathrm{d}t + \sigma_1 \left(\sqrt{1-\rho^2}\,\mathrm{d}W_1(t) + \rho\sqrt{X_2(t)}\,\mathrm{d}W_2(t)\right), \\ \mathrm{d}X_2(t) &= (b_2 - \beta_2 X_2(t))\,\mathrm{d}t + \sigma_2\sqrt{X_2(t)}\,\mathrm{d}W_2(t), \end{aligned} \tag{5.1}$$

where $\mathbf{W}$ is a 2-dimensional standard Brownian motion. The parameters satisfy b_2, β_1, β_2, σ_1, $\sigma_2 \in \mathbb{R}_+$ and $\rho \in [-1, 1]$, and the function $b_1 : \mathbb{R}_+ \to \mathbb{R}$ is chosen such that an initial term structure is fitted.

The process $\mathbf{X}_2$ models relative deviations of the surrender rate from the best estimate, and it stays non-negative, hence the surrender rate $\eta(t)$ is non-negative. The interest rate process is a mix between a Hull-White Vašíček and a Heston model.

The model satisfies our criteria. It is affine, since $\mathbf{X}$ is affine and the surrender and interest rate is an affine transformation of $\mathbf{X}$. The surrender rate is non-negative. Also, choosing no, or little, mean reversion, stress scenarios produced by the model are close to parallel shifts of the forward rates, which resembles the stress scenarios of the standard model of Solvency II.

5.1.1 Correlation

The correlation between the interest rate and the surrender rate is not in general equal to the dependency parameter ρ, which is due to the appearance of $\sqrt{X_2(t)}$ in the term $\rho\sqrt{X_2(t)}\,\mathrm{d}W_2(t)$ in the first line of (5.1). However, if we assume that $\mathrm{E}[X_2(t)] = 1$ for all t, we can calculate the correlation, using standard methods[2],

$$\mathrm{Corr}[r(t), \eta(t)] = \rho \frac{e^{(\beta_1+\beta_2)t} - 1}{\beta_1 + \beta_2} \sqrt{\frac{2\beta_1}{e^{2\beta_1 t} - 1}} \sqrt{\frac{2\beta_2}{e^{2\beta_2 t} - 1}}.$$

In the special case where $\beta_1 = \beta_2$, we get

$$\mathrm{Corr}[r(t), \eta(t)] = \rho.$$

When the parameters are chosen in Section 5.5.1 below, we see that indeed $\mathrm{E}[X_2(t)] = 1$ and $\beta_1 = \beta_2$ holds. We note that the correlation considered here is not the instantaneous correlation between the two stochastic processes $t \mapsto r(t)$ and $t \mapsto \eta(t)$, but the standard correlation between the two stochastic variables $r(t)$ and $\eta(t)$.

[2] The quantities $\mathrm{E}[r(t)]$ and $\mathrm{E}[\eta(t)]$ can be found taking expectation on the Itô representation, and solving a differential equation. The expectation $\mathrm{E}[r(t)\eta(t)]$ can be found analogously, by first finding a stochastic differential equation for the process $t \mapsto r(t)\eta(t)$.

5.2 The (Life Insurance) Product

Consider a simple savings contract with a buy-back option. The savings contract consists of a guaranteed payment of 1 at retirement at time T. There is an account at the provider with a guaranteed interest rate $\hat{r}$ until time T. The value at time t of the account is then,

$$U(t) = e^{-\hat{r}(T-t)}. \tag{5.2}$$

The owner of the savings contract can then at any time before time T surrender the contract and receive the current account value $U(t)$.

The account value $U(t)$ is not necessarily identical to the reserve (market value) of the savings contract, thus the savings contract provider has a risk. In order to best estimate the value of the account, the surrender behaviour should be taken into account. There are different ways to valuate the surrender option, see [12] and references therein, and [10]. In this paper we adopt the intensity approach, and assume that the insured surrenders with rate $\eta(t)$ at time t, i.e. in a short time interval $[t, t + \mathrm{d}t]$, the insured surrenders with probability $\eta(t)\,\mathrm{d}t$, given that surrender has not occured before time t. We adopt the life insurance setup of Section 3, and consider the state of the insured in the state space $\mathcal{J}$ consisting of the two states *alive* and *surrendered*, corresponding to Figure 2.

This savings contract is a simplified version of the product considered in Example 3.1 and the Markov model shown in Figure 1, but in order to keep the notation simple and focus on the essential parts of the formulae, the mortality modelling is omitted. As long as the mortality rate is independent of the interest and surrender rate, e.g. if it is deterministic, it is straightforward to extend the formulae to include mortality. Including mortality intuitively corresponds to reducing all payments by the probability of death. For the Solvency II studies below, we can omit mortality because the independence assumption isolates it from our dependency considerations between the interest and surrender rate.

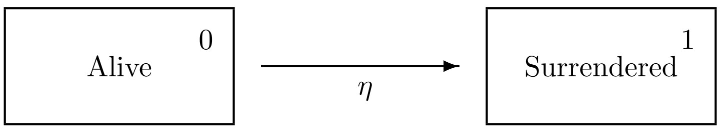

Figure 2: *Markov model for the surrender model.*

The payments of the contract consist of a single payment upon retirement at time T, and a payment upon surrender at time t of size $U(t)$. That is, the total payments $B(t)$ at time t is given by

$$\mathrm{d}B(t) = U(t)\,\mathrm{d}N_{01}(t) + 1_{(Z(t)=0)}\,\mathrm{d}\varepsilon_T(t),$$

where ε_T is the Dirac measure at T. Analogously to the calculations in Example 3.1 and Example 4.3, we find the present value at time t of the contract as

$$\begin{aligned} PV^{\mathbf{L}}(t) &= \int_t^T e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}B(s) \\ &= \int_t^T e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} U(s)\,\mathrm{d}N_{01}(s) + e^{-\int_t^T r(\tau)\,\mathrm{d}\tau} 1_{(Z(T)=0)}, \end{aligned} \tag{5.3}$$

and the market value at time t is, given the savings contract has not been surrendered,

$$\begin{aligned} V(t) &= \mathrm{E}\left[PV^{\mathbf{L}}(t) \middle| \mathcal{F}(t), Z(t)=0 \right] \\ &= \mathrm{E}\left[\int_t^T e^{-\int_t^s (r(\tau)+\eta(\tau))\,\mathrm{d}\tau} \eta(s) U(s)\,\mathrm{d}s + e^{-\int_t^T (r(\tau)+\eta(\tau))\,\mathrm{d}\tau} \middle| \mathcal{F}^{\mathbf{X}}(t) \right] \\ &= \int_t^T e^{-\int_t^s (f_t^{r:(r+\eta)}(\tau)+f_t^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_t^{\eta:(r+\eta)}(s) U(s)\,\mathrm{d}s \\ &\quad + e^{-\int_t^T (f_t^{r:(r+\eta)}(\tau)+f_t^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau}. \end{aligned} \tag{5.4}$$

Here we used Remark 4.2. The notation used is introduced in Example 4.3 above.

5.3 Solvency II

For Solvency II purposes one wants to control the risk of default, such that it is less than 99.5% during the following year. In this section we specify how to interpret this in our setup, following the reasoning of Section 1.1 in [3].

We want to find the loss after one year, which is a stochastic variable, and find quantiles in the distribution of this stochastic variable. Let $PV(t)$ denote the present value at time t of future payments of the insurance company. At time 0, the Solvency II loss can be written as

$$\mathrm{E}\left[PV(0)\,|\mathcal{F}(1)\right] - \mathrm{E}\left[PV(0)\right],$$

where the expectation is taken using the market measure, or some reserving measure. For the rest of the paper, we refer to this measure as the market measure. The last term is the value of the future payments now, and the first term is the value conditional on the following year's information, which is uncertain. For simplicity, we ignore the so-called unsystematic risk during the first year, that is, we take average of the Markov chain $\mathbf{Z}$, conditionally on the underlying intensities $\mathbf{X}$. Formally, we define the *Solvency II loss after 1 year* as

$$L = \mathrm{E}\left[PV(0)\,\middle|\mathcal{F}^{\mathbf{X}}(1)\right] - \mathrm{E}\left[PV(0)\right].$$

Both liabilities and assets must be taken into account, so the present value takes the form $PV(t) = PV^{\mathbf{L}}(t) - PV^{\mathbf{A}}(t)$, that is, the present value of the liabilities less the assets.

We consider our life insurance contract from Section 5.2. The simplest possible asset allocation is to deposit all capital in a savings account, earning the risk free interest rate. In that case, the present value of the assets is deterministic and equals the amount invested today. If the amount invested at time 0 is the value of the liabilities, (5.4), we have

$$PV^{\mathbf{A}}(0) = V(0).$$

Using this, we get the Solvency II loss,

$$\begin{aligned} L &= \mathrm{E}\left[PV^{\mathbf{L}}(0) - PV^{\mathbf{A}}(0) \middle| \mathcal{F}^{\mathbf{X}}(1)\right] - \mathrm{E}\left[PV^{\mathbf{L}}(0) - PV^{\mathbf{A}}(0)\right] \\ &= \mathrm{E}\left[PV^{\mathbf{L}}(0) \middle| \mathcal{F}^{\mathbf{X}}(1)\right] - V(0), \end{aligned}$$

and we see that the assets disappear from the formula, because they are essentially risk free. For our case, the first term is obtained from (5.3), and we get

$$\begin{aligned} &\mathrm{E}\left[PV^{\mathbf{L}}(0) \,\middle|\mathcal{F}^{\mathbf{X}}(1)\right] \\ &= \int_0^1 e^{-\int_0^s (r(\tau)+\eta(\tau))\,\mathrm{d}\tau}\eta(s)U(s)\,\mathrm{d}s \\ &\quad + e^{-\int_0^1 (r(s)+\eta(s))\,\mathrm{d}s} \int_1^T e^{-\int_1^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s)U(s)\,\mathrm{d}s \\ &\quad + e^{-\int_0^1 (r(s)+\eta(s))\,\mathrm{d}s} e^{-\int_1^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau}. \end{aligned}$$

Subtracting $V(0)$, given by (5.4), yields the loss, and we rearrange the terms slightly,

$$\begin{aligned} L = &\int_0^1 e^{-\int_0^s (r(\tau)+\eta(\tau))\,\mathrm{d}\tau}\eta(s)U(s)\,\mathrm{d}s \\ &- \int_0^1 e^{-\int_0^s (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_0^{\eta:(r+\eta)}(s)U(s)\,\mathrm{d}s \\ &+ e^{-\int_0^1 (r(s)+\eta(s))\,\mathrm{d}s} \int_1^T e^{-\int_1^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s)U(s)\,\mathrm{d}s \\ &- \int_1^T e^{-\int_0^s (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_0^{\eta:(r+\eta)}(s)U(s)\,\mathrm{d}s \\ &+ e^{-\int_0^1 (r(s)+\eta(s))\,\mathrm{d}s} e^{-\int_1^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} \\ &- e^{-\int_0^T (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau}. \end{aligned}$$

The first two lines correspond to the losses arising during the first year because of incorrect expectations of interest and surrender behaviour. The last four lines correspond to changed expectations of the future, arising because of information received during the first year. That is, the third and fourth line corresponds to changed expectations of the future about the surrender payments, and the last two lines correspond to changed expectations of the future about the payment occuring at retirement. Intuitively, the information received during the first year allows for an exact discounting during the first year, and a more precise valuation of the discounting and surrender behaviour occuring from year 1 and onwards.

The loss can be written in a simpler form. Using the notation that, for $s \leq t$, $f_t^{r:(r+\eta)}(s) = r(s)$ and $f_t^{\eta:(r+\eta)}(s) = \eta(s)$, we can write the Solvency II loss as

$$\begin{aligned} L = &\int_0^T e^{-\int_0^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s) U(s)\,\mathrm{d}s \\ &- \int_0^T e^{-\int_0^s (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_0^{\eta:(r+\eta)}(s) U(s)\,\mathrm{d}s \\ &+ e^{-\int_0^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} - e^{-\int_0^T (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau}. \end{aligned} \tag{5.5}$$

This formula gives interpretation to the Solvency II loss, which arises due to the development of the forward rates. The loss is the difference in the expected present value with the forward rates evaluated in 1 years time, and evaluated now. In practice the loss can thus be obtained by simulation of the dependent forward rates one year ahead. This is similar to how forward rates are used in finance, where the forward interest rate is simulated ahead to obtain future term structures. Here, we also simulate the dependent forward rates ahead in order to obtain the future valuation basis.

Recalling that the dependent forward rates $f_1^{r:(r+\eta)}$ and $f_1^{\eta:(r+\eta)}$ are $\mathcal{F}^{\mathbf{X}}(1)$ measurable, we can use that $\mathbf{X}$ is a Markov process and see that $f_1^{r:(r+\eta)}$ and $f_1^{\eta:(r+\eta)}$ are $r(1)$ and $\eta(1)$ measurable. Thus, the loss can be found by simulation of the underlying rates $r(s)$ and $\eta(s)$ for $0 \leq s \leq 1$. The simulation must be done under the real world probability measure. This is opposed to the market, or reserving, measure, that was used to find the loss. In this paper, we assume for simplicity that the two measures are identical, and do not adapt a change of measure approach, relieving us from discussions of preservation of the Markov property during measure changes.

5.4 Hedging Strategy with a Continuously Paid Coupon Bond

In practice, an insurer tries to hedge the interest rate risk, thereby reducing the loss significantly. We consider a simple static hedging strategy, in a bond with continuous coupon payments of the form,

$$c(t) = e^{-\int_0^t f_0^{\eta:\eta}(\tau)\,\mathrm{d}\tau} f_0^{\eta:\eta}(t) U(t),$$

for $t \in (0, T)$, and a final payment at time T of

$$C(T) = e^{-\int_0^T f_0^{\eta:\eta}(\tau)\,\mathrm{d}\tau}.$$

For more details, see e.g. [12]. This corresponds to the expected payments of the life insurance contract, conditional on the interest rate. We can associate a payment process $\mathbf{A}^{\text{bond}}$ with the bond, given by $\mathrm{d}A^{\text{bond}}(t) = c(t)\,\mathrm{d}t + C(t)\,\mathrm{d}\varepsilon_T(t)$. The present value of future payments associated with the bond is then,

$$\begin{aligned} PV^{\text{bond}}(t) &= \int_t^T e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A^{\text{bond}}(s) \\ &= \int_t^T e^{-\int_t^s \left(r(\tau)+f_0^{\eta:\eta}(\tau)\right)\mathrm{d}\tau} f_0^{\eta:\eta}(s)U(s)\,\mathrm{d}s + e^{-\int_0^T \left(r(\tau)+f_0^{\eta:\eta}(\tau)\right)\mathrm{d}\tau}. \end{aligned}$$

This hedging strategy is the mean-variance optimal static hedging strategy when interest and surrender are independent. If there is a correlation between the interest and surrender rate, this strategy is not optimal. The mean-variance optimal static hedging strategy is in that case more complicated. These considerations are for simplicity omitted in this paper, and deferred for future studies.

In the case of dependence, the price of the hedging bond is smaller than the value of the liabilities, so the expected present value of the bond payments $\mathbf{A}^{\text{bond}}$ is less than of the payments from the savings contract $\mathbf{B}$. We choose to put this excess capital, which is given as

$$K = E\left[PV^{\mathbf{L}}(0) - PV^{\text{bond}}(0)\right],$$

in the bank account. For the assets, we thus have present value at time 0

$$PV^{\mathbf{A}}(0) = PV^{\text{bond}}(0) + K.$$

We note that the sign of the payments $\mathbf{A}^{\text{bond}}$ is opposite of $\mathbf{B}$, where the latter are payments to the insured and the former are payments to the insurer. Considering the life insurance contract and the hedging strategy together, we obtain a Solvency II loss,

$$\begin{aligned} L &= \mathrm{E}\left[PV^{\mathbf{L}}(0) - PV^{\mathbf{A}}(0)\middle|\,\mathcal{F}^{\mathbf{X}}(1)\right] - \mathrm{E}\left[PV^{\mathbf{L}}(0) - PV^{\mathbf{A}}(0)\right] \\ &= \mathrm{E}\left[\int_0^T e^{-\int_0^s r(\tau)\,\mathrm{d}\tau}\left(\mathrm{d}B(s) - \mathrm{d}A^{\text{bond}}(s)\right)\middle|\,\mathcal{F}^{\mathbf{X}}(1)\right] - K \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 e^{-\int_0^s r(\tau)\,\mathrm{d}\tau} \left(e^{-\int_0^s \eta(\tau)\,\mathrm{d}\tau} \eta(s) - e^{-\int_0^s f_0^{\eta:\eta}(s)\,\mathrm{d}s} f_0^{\eta:\eta}(s) \right) U(s)\,\mathrm{d}s \\
&\quad + e^{-\int_0^1 (r(s)+\eta(s))\,\mathrm{d}s} \Bigg(\int_1^T e^{-\int_1^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s) U(s)\,\mathrm{d}s \\
&\qquad + e^{-\int_1^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} \Bigg) \\
&\quad - e^{-\int_0^1 (r(s)+f_0^{\eta:\eta}(s))\,\mathrm{d}s} \Bigg(\int_1^T e^{-\int_1^s (f_1^{r:r}(\tau)+f_0^{\eta:\eta}(\tau))\,\mathrm{d}\tau} f_0^{\eta:\eta}(s) U(s)\,\mathrm{d}s \\
&\qquad + e^{-\int_1^T (f_1^{r:r}(\tau)+f_0^{\eta:\eta}(\tau))\,\mathrm{d}\tau} \Bigg) - K \\
&= \int_0^T e^{-\int_0^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s) U(s)\,\mathrm{d}s \\
&\quad - \int_0^T e^{-\int_0^s (f_1^{r:r}(\tau)+f_0^{\eta:\eta}(\tau))\,\mathrm{d}\tau} f_0^{\eta:\eta}(s) U(s)\,\mathrm{d}s \qquad (5.6) \\
&\quad + e^{-\int_0^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} - e^{-\int_0^T (f_1^{r:r}(\tau)+f_0^{\eta:\eta}(\tau))\,\mathrm{d}\tau} - K,
\end{aligned}
$$

Similar to (5.5), for $s \leq t$, the notation that $f_t^{r:(r+\eta)}(s) = f_t^{r:r}(s) = r(s)$ and $f_t^{\eta:(r+\eta)}(s) = \eta(s)$ is used for the last equality. When there is independence the bond value is the same as the value of the savings contract and $K = 0$. When there is dependence we have $K > 0$, which ensures that $\mathrm{E}[L] = 0$.

5.5 Numerical Results

In this section we numerically show some consequenses of modelling interest and surrender as positively correlated processes. First, the model is specified by choosing a set of parameters, partly inspired by the stress levels in the Solvency II Standard Formula. With this model, we examine the consequenses for the balance sheet value of the liabilities, and the level of the Solvency II capital requirement, that is, the liabilities in 1 year's time.

For the Solvency II capital requirement, in practice in the industry, when there is no hedging, most of the risk is interest rate risk. Luckily, both in theory and practice, a lot of this can be hedged by e.g. buying bonds. For the numerical illustrations of the Solvency II capital requirement, we consider two different strategies for the assets, corresponding to the two strategies considered in Section 5.3 and Section 5.4, respectively. First, we consider the case where the interest rate risk is not hedged, and all assets are accumulated by the risk free interest rate. Second, we consider the case where the insurer tries to hedge the interest rate risk, and performs a static hedge.

5.5.1 Parameters

The numerical examples with the model (5.1) are carried out for different level of correlation, namely $\rho \in \{0, 0.3, 0.7\}$. Also, we consider two different guaranteed interest rates, namely $\hat{r} \in \{1\%, 4\%\}$. This corresponds to a low interest rate, which could be for a newly issued policy, and a high interest rate, which could be for a policy issued years ago, when the interest rate level was higher. We note that the base deterministic surrender rate η^0 corresponds to a person aged 40, thus with $T = 25$, the contract ends at age 65.

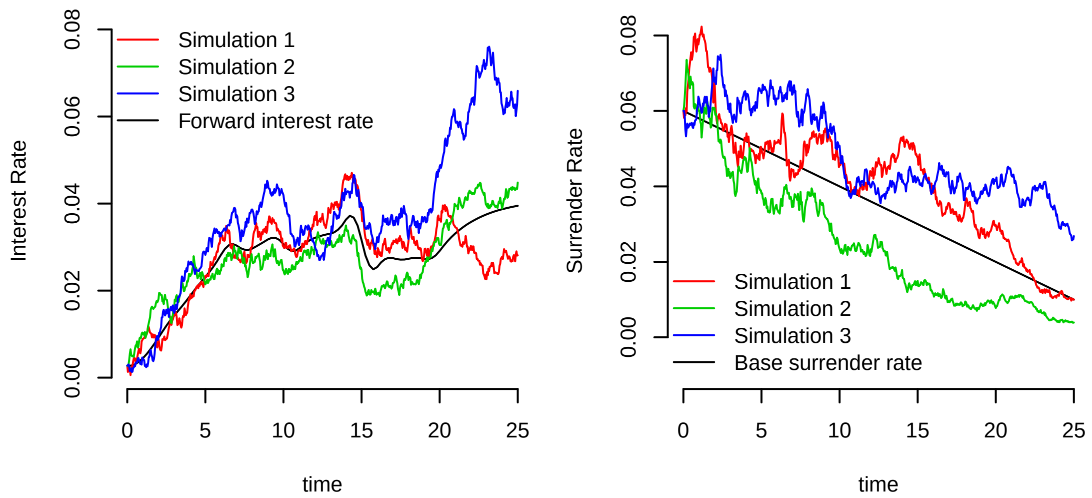

Figure 3: *Illustrative realisations of the interest rate (left) and the surrender rate (right), with* $\rho = 0.7$.

The parameters chosen for the interest and surrender rates are listed in Table 1, and in Figure 3 some realisations of the interest and surrender rates are shown. The initial value $X_1(0)$ and function $b_1(t)$ are chosen such that the term structure provided by the Danish FSA at August 17, 2012 is matched. Let $f^{\mathrm{FSA}}(t)$ denote the forward rate provided by the Danish FSA. Then the parameters X_1 and b_1 are fitted such that

$$\mathrm{E}\left[e^{-\int_0^t r(s)\,\mathrm{d}s}\right] = e^{-\int_0^t f^{\mathrm{FSA}}(s)\,\mathrm{d}s},$$

for all $t \geq 0$. The parameters of the model correspond to the measure used for valuating the market value of the life insurance liabilities. Thus, with respect to the interest rate it is the risk neutral measure. For simplicity, we assume that this measure equals the real world probability measure.

β_1	0.02	b_2	0.02	$\eta^0(t)$	$0.06 - 0.002 \cdot t$
σ_1	0.005	β_2	0.02	$X_2(0)$	1
		σ_2	0.15		

Table 1: *Parameters for correlated interest and surrender modelling. The initial value $X_1(0)$ and the function $b_1(t)$ are chosen such that the interest rate model matches the term structure provided by the Danish FSA for valuating life insurance liabilites, at August 17, 2012.*

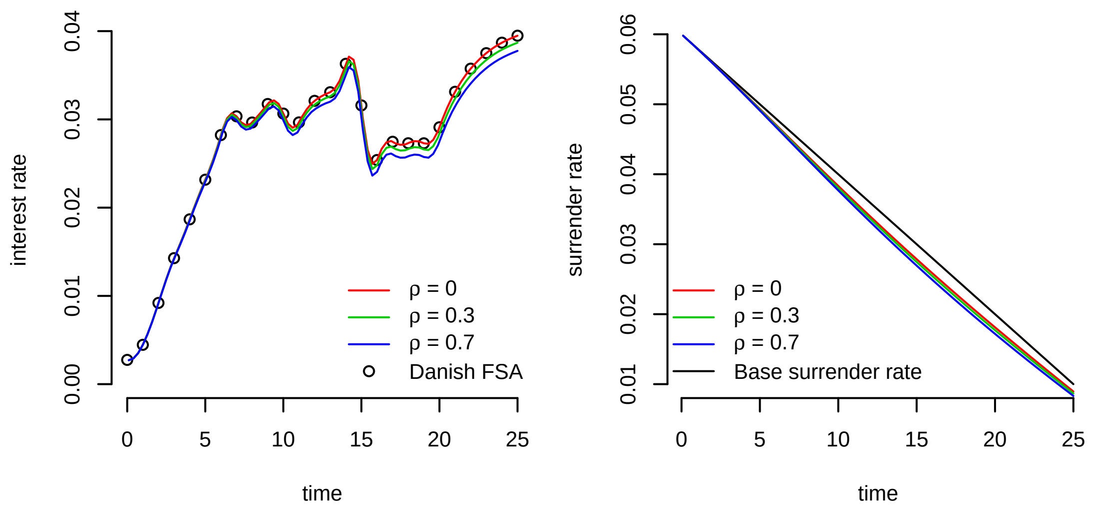

Figure 4: *Dependent forward rates. Left: for the interest rate, $f_0^{r:(r+\eta)}(t)$. Right: for the surrender rate, $f_0^{\eta:(r+\eta)}(t)$. The dependent forward rates are shown for different values of ρ. The forward interest rate extracted from the Danish FSA at August 17, 2012 is also shown, as well as the base deterministic surrender rate η^0. Higher values of ρ lead to lower values of the forward rates, corresponding to less discounting.*

5.5.2 Dependent Forward Rates

In Figure 4, the dependent forward rates are shown. They are calculated by solving the differential equations (2.3) and (2.5) numerically. For the interest rate, the forward interest rate supplied by the Danish FSA, f^{FSA}, is shown as well. We see that for the case $\rho = 0$ the dependent forward interest rate f_0^r is identical to the forward rate provided by the Danish FSA. This is as expected, since in the case $\rho = 0$ the interest rate and surrender rate are independent, and in this case the dependent forward rates are equal to the usual forward rates. For a positive correlation, the dependent forward rates are smaller. This is because the stochastic variable, $e^{-\int_0^t (r(s)+\eta(s))\,\mathrm{d}s}$, which is used to construct the dependent forward rates, has a heavier tail when the correlation is strictly positive, due to the exponential function. Intuitively, there is less diversification between the interest and surrender rate.

For the surrender rate, the basic deterministic surrender rate η^0 is shown as well as the dependent forward rates. Even though $\mathrm{E}\left[\eta(t)\right] = \eta^0(t)$, we see that the dependent forward rates are systematically lower than η^0. This is due to Jensens inequality, and to see this, consider the case $\rho = 0$, where we get,

$$\begin{aligned} e^{-\int_0^t (f_0^r(s)+f_0^\eta(s))\,\mathrm{d}s} &= \mathrm{E}\left[e^{-\int_0^t (r(s)+\eta(s))\,\mathrm{d}s}\right] \\ &= \mathrm{E}\left[e^{-\int_0^t r(s)\,\mathrm{d}s}\right]\mathrm{E}\left[e^{-\int_0^t \eta(s)\,\mathrm{d}s}\right] \\ &> \mathrm{E}\left[e^{-\int_0^t r(s)\,\mathrm{d}s}\right] e^{-\int_0^t \mathrm{E}[\eta(s)]\,\mathrm{d}s} \\ &= e^{-\int_0^t f_0^r(s)\,\mathrm{d}s} e^{-\int_0^t \eta^0(s)\,\mathrm{d}s}, \end{aligned}$$

for $t > 0$, using that the usual forward rate is identical to the dependent forward rate for $\rho = 0$. From this inequality, we obtain,

$$f_0^\eta(t) < \eta^0(t),$$

which is what was observed as the red and black lines in Figure 4. If there is a positive correlation, the dependent forward surrender rate, $f^{\eta:(r+\eta)}$, is even smaller, similar to the observation for the interest rates.

5.5.3 Market Value

The market value at time 0, $V(0)$ from (5.4), can be calculated, solving the integral numerically. For this, first use (4.1) to get

$$V(t) = \int_t^T e^{\phi(t,s)+\psi(t,s)^\top X(t)} f_t^{\eta:(r+\eta)}(s)U(s)\,\mathrm{d}s + e^{\phi(t,T)+\psi(t,T)^\top X(t)},$$

which is easier to handle from a computational point of view, because the functions ϕ and ψ are obtained in the process of calculating the dependent forward rates $f^{r:(r+\eta)}$ and $f^{\eta:(r+\eta)}$ when solving (2.3) and (2.5). The market value $V(0)$, dependent upon the guaranteed interest rate $\hat{r}$ and the correlation ρ, is shown in Table 2. The market values can be compared to the value of the policyholders account which is paid out on surrender. This is given by (5.2), calculated using the guaranteed interest rate. The value at time 0 is presented in Table 3.

The market value without surrender modelling, calculated setting the surrender rate equal to zero, is 0.5037. It is independent of the guaranteed interest rate. From Table 2 it is seen that when we include surrender modelling the market value is somewhere between the value of the policyholders account and the market value calculated without surrender modelling.

		$\hat{r}$	
		4%	1%
	0	0.4567	0.6167
ρ	0.3	0.4595	0.6191
	0.7	0.4631	0.6222

Table 2: *Market value at time 0, $V(0)$, of the life insurance contract. The value is shown using three different correlations, corresponding to three different sets of dependent forward rates, red, green and blue from Figure 4. Two different levels of guaranteed interest rate, $\hat{r}$, is used, which leads to different surrender payouts $U(t)$.*

$\hat{r}$	
4%	1%
0.3679	0.7788

Table 3: *Initial value of the policyholders account, $U(0)$. For the high guaranteed interest rate (4%), the value is lower than the market value from Table 2. For the low guaranteed interest rate (1%), the value is higher than the market value.*

For both cases of guaranteed interest rates, the market value increases with correlation. When we discussed the dependent forward rates in Section 5.5.2, we saw that the dependent forward rates decrease with increasing correlation, which is basically due to the convexity of the exponential function and Jensen's inequality. A smaller dependent forward interest rate leads to an increasing market value. For the surrender rate, it is more complicated. For the case of a guaranteed interest rate of 4%, an increase in the dependent forward surrender rate leads to a decrease in the market value, because the market value come closer to the value paid out on surrender. For the case of a guaranteed interest rate of 1%, the same argument tells us that an increase in the dependent forward surrender rate instead leads to an increasing market value. We see that the effect of the decreasing dependent forward interest rate is largest, and in total, for both levels of guaranteed interest rate, the market value increases when the correlation increases.

5.5.4 Solvency II

We examine the effect on the Solvency II capital requirement with two different strategies for the assets. The first strategy is no hedging and the second strategy is a simple static hedging strategy. This corresponds to the two strategies discussed in Section 5.3 and Section 5.4, respectively. For the first strategy, where all assets are invested in the bank account, the Solvency II loss is given by (5.5). For the second strategy, where the interest rate risk is hedged statically in a bond with continuous payments, the Solvency II loss is given by (5.6).

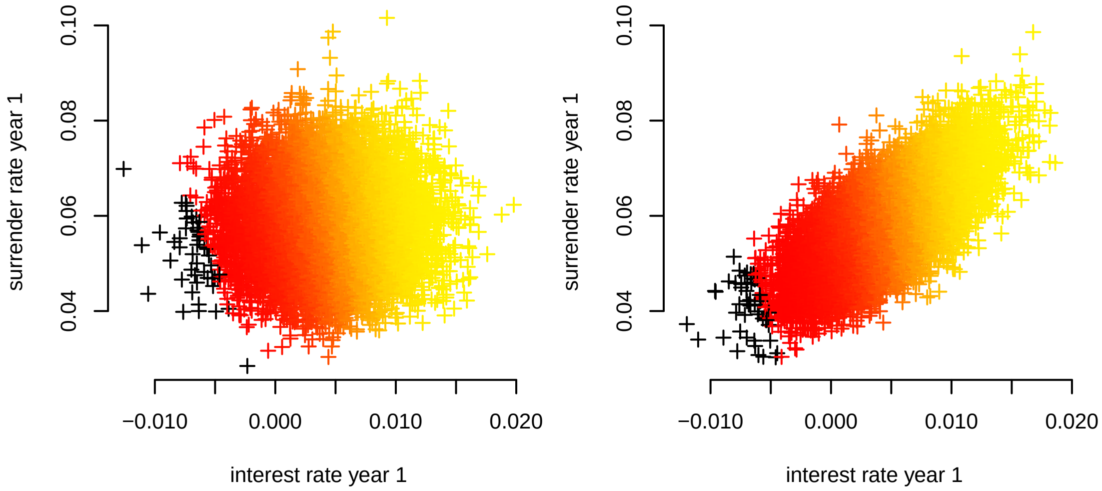

Figure 5: *Guaranteed interest rate 4%. Plot of the interest and surrender rate simulations after 1 year in the case without any hedging strategy and correlation* $\rho = 0$ *(left) and* $\rho = 0.7$ *(right). The color of a mark indicates the Solvency II loss* (5.5)*, where a darker color is a higher loss, and black colors are losses beyond the 99.5% quantile.*

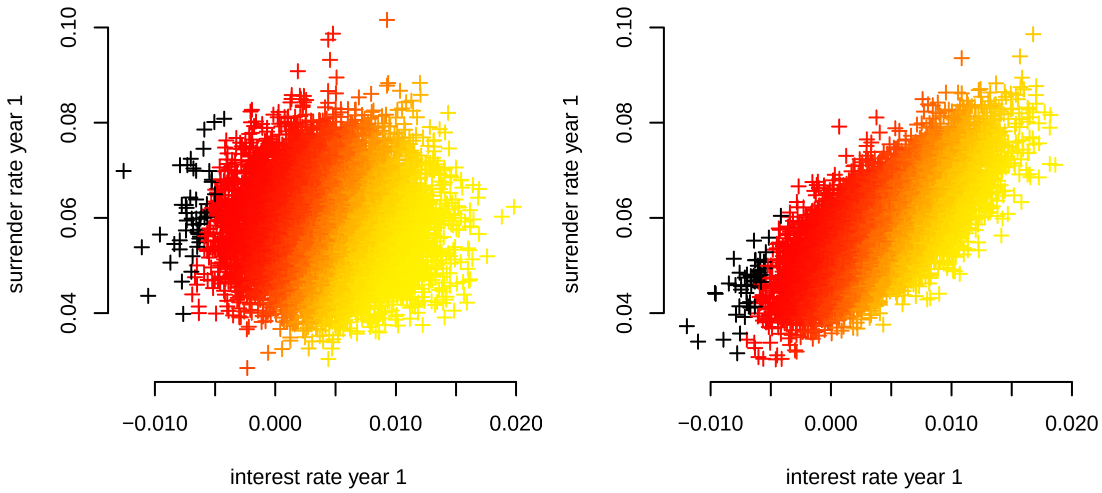

Figure 6: *Guaranteed interest rate 1%. Plot of the interest and surrender rate simulations after 1 year in the case without any hedging strategy and correlation* $\rho = 0$ *(left) and* $\rho = 0.7$ *(right). The color of a mark indicates the Solvency II loss* (5.5)*, where a darker color is a higher loss, and black colors are losses beyond the 99.5% quantile.*

		No Hedge		Hedge	
		$\hat{r}$		$\hat{r}$	
		4%	1%	4%	1%
	0	0.069	0.077	0.014	0.025
ρ	0.3	0.072	0.072	0.015	0.028
	0.7	0.078	0.060	0.017	0.029

Table 4: *Simulated Solvency II loss. Without hedging it is given by* (5.5) *and with the hedging strategy it is given by* (5.6). *Applying an interest hedging strategy significantly lowers the Solvency II loss. Also, modelling correlation between interest and surrender has a significant impact on the Solvency II loss.*

In Table 4 the Solvency II loss for the different cases of hedging strategy, guaranteed interest rate risk and correlation is presented. It is immediately seen, that trying to hedge the interest rate risk by applying the simple hedging strategy significantly reduces the Solvency II loss.

For the case of no hedging strategy, we see two different correlation effects. When the guaranteed interest rate is 4%, a higher correlation means a higher Solvency II loss, because a decrease in interest and surrender rate both increase the present value of the contract payments. This is depicted in Figure 5, where we see that the loss increases with both decreasing interest and decreasing surrender. A higher correlation means that the probability of simultaneous drops in the interest and surrender rate occurs simultaneously, which can be seen at the right graph in Figure 5. When the guaranteed interest rate is instead 1%, a decrease in the surrender rate now means that Solvency II loss decrease, which can be seen in Figure 6. Introducing a correlation leads to less observations with decreasing interest and increasing surrender, thus leading to more diversification and reducing the Solvency II loss. This can be seen in the right graph of Figure 6.

6 Conclusion

In this paper we review theory on general affine processes which sets the basis for the application in life insurance valuation and risk management. This allows us to introduce so-called dependent forward rates. They are compared to other forward rate definitions, and some desired properties about the dependent forward rates are highlighted. In particular, as is seen in Section 4.1.1, the dependent forward rates meet some of the critique of forward rates raised in [14]. However, a full answer is not reached, and it is open for further research whether the concept of forward rates in life insurance is fruitful beyond being a convenient representation for the quantities needed for calculation of certain life insurance liabilites under a stochastic intensity assumption.

In the second part of the paper, we apply the theory of the general affine processes and the dependent forward rates. A specific model for surrender modelling is proposed, where the interest and surrender rate is positively correlated. The surrender rate in this model is non-negative. We consider a simple life insurance like savings product with a buy-back option. The dependent forward rates are calculated for different correlations, and we see that they are decreasing with increasing correlation. This in part has the effect that the market value is increasing with correlation, since in part, this means we in practice use a smaller interest rate for discounting. We also consider the Solvency II capital requirement in the form of formulae for the value-at-risk in a one-year time horison. In particular we obtain the formula (5.5), where we see that the loss is given through the difference of the expected present value valuated with the dependent forward rates in 1 year and the current dependent forward rates. We calculate the actual Solvency II capital requirement in our example, with and without a simple static hedging strategy for the interest rate risk, and see how the introduction of correlation can both increase and decrease the Solvency II capital requirement in our example, depending on the guaranteed interest rate.

Acknowledgements

I would like to thank my ph.d.-supervisors Thomas Møller and Mogens Steffensen for their guidance and support during this work. I would also like to thank two anonymous referees for comments and proposals that improved the paper.

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A Forward Mortality Rate for Term Insurances not-so-well Defined

Consider a interest and mortality rate model $(r(t), \mu(t))$. This give us a set of dependent forward rates, and a forward mortality rate for pure endowment. Assume that the following assumptions hold.

Assumption A.1. *Let a model for the interest and mortality rates $r(t)$ and $\mu(t)$ be given. The assumptions are,*

1. $h_t^{\mathrm{pe}}(s) > 0$ *for all* $s > t$.
2. h_t^{pe} *is bounded from below for some timepoint, i.e. there exist* $\varepsilon > 0$ *and* $t_0 > 0$ *such that* $h_t^{\mathrm{pe}}(s) > \varepsilon$ *for all* $s > t_0$.
3. *The forward interest rate is greater than the dependent forward rate for the interest,* $g_t(s) > f_t^r(s)$, *for all* $s > t$.

It is indeed possible to construct models where these assumptions hold, and they will hold for most models when there is a positive correlation between the interest rate and mortality rate. The first two assumptions state that the forward mortality rate for pure endowment is positive and bounded below from some time, which is satisfied in reasonable models. The third assumption usually holds when there is a positive correlation between the interest and mortality rate.

The forward mortality rate for pure endowment, $h_t^{\mathrm{pe}}(s)$, present in the assumptions, is not the object of interest in this example. In view of (4.8), it can be thought of as a placeholder for $f_t^r + f_t^\mu - g_t$.

Proposition A.2. *Under Assumption A.1, there exists a $T > 0$ such that the forward mortality rate for term insurances $h_t^{\mathrm{ti}}(s)$ given by* (4.10) *does not exist for $s > T$.*

Proof. Combining (4.10) and (4.3), and then using (4.8) twice, we get that

$$\begin{aligned} e^{-\int_t^T h_t^{\mathrm{ti}}(s)\,\mathrm{d}s} h_t^{\mathrm{ti}}(T) &= e^{-\int_t^T \left(f_t^r(s)+f_t^\mu(s)-g_t(s)\right)\mathrm{d}s} f_t^\mu(T) \\ &= e^{-\int_t^T h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} \left(h_t^{\mathrm{pe}}(T) + g_t(T) - f_t^r(T)\right), \end{aligned}$$

and by integration we find

$$e^{-\int_t^T h_t^{\mathrm{ti}}(s)\,\mathrm{d}s} = 1 - \int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} \left(h_t^{\mathrm{pe}}(\tau) + g_t(\tau) - f_t^r(\tau)\right) \mathrm{d}\tau. \tag{A.1}$$

Since the left hand side must be positive for any T, we conclude that the condition

$$\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} \left(h_t^{\mathrm{pe}}(\tau) + g_t(\tau) - f_t^r(\tau)\right) \mathrm{d}\tau < 1 \tag{A.2}$$

is necessary for the forward mortality rate for term insurances to be well-defined.

Under the first assumption, the forward mortality rate for pure endowment, h_t^{pe}, defines a distribution in a two-state Markov chain, and we recognise the integral $\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} h_t^{\mathrm{pe}}(\tau)\,\mathrm{d}\tau$ as a probability: Let Z be a stochastic variable that denotes the lifetime in a survival model where death occurs with rate $h_t^{\mathrm{pe}}(s)$ at time s. Then

$$\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} h_t^{\mathrm{pe}}(\tau)\,\mathrm{d}\tau = P(Z \le T \mid Z > t).$$

Also, under the second assumption the probability converges to 1,

$$P(Z \le T \mid Z > t) \to 1 \text{ for } T \to \infty.$$

Consider now (A.2). Under the third assumption, $g_t(s) > f_t^r(s)$ for all $s > t$, there exists $\varepsilon > 0$ and $T^* \ge t$ such that

$$\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} \left(g_t(\tau) - f_t^r(\tau)\right) \mathrm{d}\tau > \varepsilon,$$

for all $T > T^*$. This allows us to conclude, for a $T > T^*$ large enough, such that $P(Z \le T \mid Z > t) > 1 - \varepsilon$, that

$$\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} \left(h_t^{\mathrm{pe}}(\tau) + g_t(\tau) - f_t^r(\tau)\right) \mathrm{d}\tau > P(Z \le T \mid Z > t) + \varepsilon > 1.$$

This contradicts (A.2), and the forward mortality rate for term insurances does not exist. □

We give an example of a model satisfying Assumption A.1. Let the 2-dimensional process $\mathbf{X}$ satisfy

$$\begin{aligned} \mathrm{d}X_1(t) &= (1 - X_1(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W_1(t), \\ \mathrm{d}X_2(t) &= (1 - X_2(t))\,\mathrm{d}t + \sigma\lambda\,\mathrm{d}W_1(t) + \sigma\sqrt{1-\lambda^2}\,\mathrm{d}W_2(t), \end{aligned}$$

with $X(0) = (1,1)^\top$. Let the interest rate and mortality rate be given by

$$\begin{aligned} r(t) &= r_0 X_1(t), \\ \mu(t) &= \mu^\circ(t) + X_2(t) - 1, \end{aligned}$$

with parameters $\lambda = 0.8$, $\sigma = 0.07$ and base mortality

$$\mu^\circ(t) = 5 \cdot 10^{-4} + 7.5858 \cdot 10^{-5} \cdot 1.09144^{50+t}.$$

That this model satisfies Assumption A.1 can be shown by solving relevant differential equations.